Abstract
Imbalanced international trade and the limited number of containers in the maritime transportation industry have resulted in considerable costs and decreased profits for shipping firms. To manage these issues, this study utilizes price management techniques to control empty container repositioning in a transportation system. In this system, there is a firm providing transportation services between two ports in both directions. Each direction has its own uncertain potential demand that can be affected by the prices and the quality set by the firm. We develop a mathematical formulation to set the quality and prices of the service in order to maximize the firm’s profit. Due to demand uncertainty, we analyze different states of demand realization and discuss optimal policies for each state. These policies are then analyzed to develop a robust pricing and quality setting strategy under uncertainty. In addition, in order to easily implement the developed strategy, we propose an algorithm to convert the obtained results into a practical procedure that can be implemented by the shipping firm. Lastly, we investigate the impact of several factors on the firm’s profit and demonstrate that not only the firm’s profit but also the imposed regret depend on both internal and external factors such as the potential demands, price sensitivity, empty and loaded movement cost, and uncertainty control parameter.

Keywords:
Pricing Strategy; Service Quality; Maritime Transportation; Mathematical Formulation; Uncertain Demand; Robust Optimization

1. Introduction
Maritime transportation is one of the main modes of international shipping transport and it has been growing steadily over the last twenty years (Li et al., 2007; Feng and Chang, 2008; Dong and Song, 2009; Chen and Zeng, 2010; Zhang et al., 2014; Chen et al., 2016; Zheng et al., 2017; Lee and Song, 2017; Yu et
al., 2019; Zhang and Zhang, 2020). According to a recent report on maritime transportation published by
the United Nations, 90% of the world’s trade (10.7 billion tons of goods) was transported by sea in 2018.
This report also pointed out that the world’s seaborne trade volume saw a 4% annual growth in 2018, which
was a faster growth rate than the five years previous to 2018 (UNCTAD, 2019). In addition, the seaborne
trade industry is forecasted to grow 3.8% between 2018 and 2023 (Transport Canada, 2019), and it is
important to note that container shipping is the fastest growing sector of this industry (Rohacs and
Simongati 2007; Paulauskas and Bentzen 2008; Liu et al. 2009; Su and Wang 2009), having increased from
62 million TEUs in 2000 to 152 million TEUs in 2018 (UNCTAD, 2019).

Firms providing maritime shipping services usually use containers to ship materials and goods from
shippers in export-oriented areas to consignees in import-oriented areas. These firms usually provide
shipping services between two specific continents or regions, such as Asia-Europe service and Asia-
America service (Feng and Chang, 2008). The imbalance in international trade between different continents
or regions, which arises from different economic needs, results in different supply and demand patterns for
empty containers at different ports (Cheung and Chen, 1998; Zhou and Lee, 2009; Yun et al., 2011; Zheng
et al., 2017). As a result, empty containers accumulate at demand regions, while the supply regions are
often faced with a shortage of empty containers. Since container availability is essential for shipping firms
to meet customer demands, shipping firms need to reposition empty containers from demand regions to
supply areas to ensure the continuity of shipping activities (Yun et al., 2011). This type of movement
accounts for about 30% of the movement of all containers, and about 20% of global port handling (Rodrique
and Notteboom, 2015; Wang and Tanaka, 2016; Hosseini and Sahlin, 2019). Therefore, Empty Containers
Repositioning (ECR) imposes undesirable expenses on shipping firms and has become a major issue for
them.

In order to manage ECR, some firms utilize foldable containers. For instance, Zhang et al. (2020)
developed a mathematical model to investigate the potential of using foldable containers to improve empty
container repositioning in river–sea intermodal transport. Their results show that employing foldable
containers in empty container repositioning can help utilize vessel space more effectively and decrease the
total cost for container shipping companies. Pricing is another useful approach that shipping firms have utilized. Currently, pricing has been used as an appropriate technique for shipping companies because traditional cost-plus pricing approaches, which are deeply rooted in the industry, are ineffective. Using the pricing technique can increase margins by 2 to 7% in as little as 12 months, and yield a 200 to 350% return on investment (Quintiq, 2016). In addition, pricing in the transportation industry is not only useful for revenue management, as in many other industries; it is also an appropriate tool for adjusting the realized demand and the corresponding empty container repositioning flow (Topaloglu and Powell, 2007; Zhou and Lee, 2009; Chen et al., 2016; Zheng et al., 2017).

The price determined by the shippers should cover all their shipping costs, including port costs. Although all ports attempt to render productive handling services to their customers, there are a number of ports that deliver benefits to their customers that are superior to their competitors. Providing these value-added services intuitively means extra costs for these ports. Therefore, the port costs depend upon two main factors: 1) cargo handling service capabilities, and 2) the benefits they are capable of delivering (UNCTAD, 1992). Relevant studies demonstrate that offering value-added services in addition to productive cargo-handling services increases the shipper’s success rate in customer attraction (Huang et al., 2013; Han et al., 2017). Note that, this factor might have a twofold effect on empty repositioning in the transportation industry. On one side, providing value-added service increases the cost of service and consequently, its price; on the other side, it alters the realized demand and its corresponding empty repositioning cost. Hence, choosing an appropriate level of service quality and value for the price of shipping services can assist shipping firms with adjusting their realized demand and the corresponding ECR cost, thereby maximizing their earned profits. To achieve optimal prices and level of quality, shipping firms encounter two challenges: 1) determining how the service price and quality affect the realized demand and its corresponding ECR; and 2) understanding how they can determine the optimal value of prices and the level of service quality simultaneously. In addition to these challenges, there are several factors that make the future of maritime trade demand uncertain (Cariou, 2020). Some of these factors include the level of integration and regionalization of the global economy, the prospects on the world population and the GDP per capita.
Although demand uncertainty increases the complexity of the decision-making process, its impact should be captured and investigated since overlooking this uncertainty may lead to significant regret or losses for the shipping firms.

To handle these issues and challenges, this study aims to formulate the pricing problem mathematically and to define an optimal strategy to determine the service price and quality for a shipping firm in a monopoly system. It is worthwhile to note that although the current container shipping market is not a monopoly, and there are several companies (e.g. APM-Maersk, MSC\textsuperscript{ii}, and COSCO\textsuperscript{iii}) that compete, investigating pricing strategies in a monopoly market is still useful. As the obtained results can assist these companies with forming strategic alliances (Zhou and Lee, 2009), which have become more common over the last 5-6 years (Marinelli, 2019). To this end, we focus on a transportation service problem with two regions, and therefore, two directions with their own potential demands. These potential demands could be balanced or imbalanced, and consequently, they result in different realized demands in each direction. We develop pricing and quality setting strategies for two cases: deterministic potential demands and uncertain potential demands. In the case of uncertainty, the potential demand is assumed to vary within a range. Since there are a limited number of containers to transport cargo between these two regions, the firm must reposition the empty containers from the region that has more supply to the region that has more demand, in the case of imbalanced realized demand.

The remainder of this paper is organized as follows. Section 2 reviews related literature to position this study among existing works and reveal its novel aspects. Section 3 defines the problem in detail, and identifies parameters and decision variables. Due to the complexity of the problem under investigation, we first analyze it in a deterministic environment and develop an algorithm to determine the service price and quality in Section 4. Then, Section 5 utilizes the obtained results from Section 4 to develop a strategy for cases with uncertain potential demand. Afterwards, Section 6 conducts numerical analyses to investigate the impact of different parameters on the firm’s profit and to generate managerial insights. Lastly, conclusions and future research directions are presented in Section 7.
2. Literature Review

Empty Container Repositioning, which is a key challenge in maritime transportation systems, has been widely investigated in the literature since as early as the 1970s (e.g., Ermol’ev et al., 1976). Throughout this time, several research studies have been conducted using different quantitative and management science techniques to minimize empty container repositioning costs in containerized transportation systems. The most investigated areas have been: inventory management (e.g., Li et al., 2007; Dong & Song, 2007; Feng & Chang, 2008; Wang & Tang, 2010; Lofstedt et al., 2011; Song & Dong, 2011, 2012), planning horizon management (e.g., Cheung & Chen, 1998; Choong et al., 2002; Jansen et al., 2004), distribution management (e.g., Gao, 1994; Shen & Khoong 1995; Coslovich et al., 2006; Lu et al., 2010a, b; Zeng et al., 2010; Zurheide & Fischer, 2012; Chao & Yu, 2012; Wang et al., 2013; Jeong et al., 2018; Shintani et al., 2019), and price management (e.g., Topaloglu & Powell, 2007; Zhou & Lee, 2009; Zheng et al., 2017; Yu et al., 2019; Zhang & Zhang, 2020).

Many of the earlier studies focused on deterministic systems. For instance, Ermol’ev et al. (1976) proposed a model to obtain the optimal movement of empty containers in a simple given network considering the cost of shipping and container rentals. Gao (1994) developed a deterministic model to make decisions about two issues: container fleet sizing and reallocation. His proposed two-stage model investigates both the long-term and mid-term capital investment in container purchasing and the operational costs of empty container leasing, allocation, and storage. Gendron and Crainic (1995) also considered a deterministic system and developed a minimum-cost network flow model to address a multi-commodity location problem with balancing requirements. Developing a deterministic model for dynamic allocation, and reusing the containers considering their long-term and operational costs, is another issue that has been investigated extensively in the literature (e.g., Shen & Khoong, 1995; Olivo et al., 2005; Jula et al., 2006; Shintani et al., 2007; Lofstedt et al., 2011; Song & Dong, 2012; Wang et al., 2013; Zheng et al., 2015). These studies utilized several modelling and solution approaches, such as network optimization (Shen & Khoong, 1995; Shintani et al., 2007; Shintani et al., 2019), Lagrangian relaxation (Coslovich et al., 2006), and Integer programming (Olivo et al., 2005; Jula et al., 2006; Lofstedt et al., 2011; Song & Dong, 2012;
Wang et al., 2013; Zheng et al., 2015; Jeong et al., 2018), to handle container allocation to available demand nodes. Although these deterministic models have their own advantages and can assist decision makers with improving long-term decisions, they do not capture the important stochastic characteristics of demand and/or supply.

Since the 1990s, some researchers have focused on the stochastic characteristics of the problem due to the weakness of deterministic models (e.g., Crainic et al., 1993; Lai et al., 1995; Cheung & Chen, 1998). Crainic et al. (1993) developed several dynamic and stochastic formulations to investigate the possibility of leasing in and off in a multi-commodity system with different container types. Lai et al. (1995) evaluated several allocation policies under uncertain demand to reduce operational costs and prevent empty container shortage. In addition, Cheung and Chen (1998) investigated a dynamic container allocation problem under demand uncertainty to make decisions about both ECR and the number of lease containers needed at a port.

Li et al. (2004) also considered an uncertain environment. They investigated demand uncertainty in a single port system and proposed a dynamic programming model to manage ECR costs. They later extended their research to a multi-port system (Li et al., 2007). Similar to this work, Lam et al. (2007) proposed a dynamic programming model in a two-port two voyages system with stochastic demand. In addition, Song and Earl (2008) considered demand uncertainty in a two-depot system and Song and Zhang (2010) investigated demand uncertainty in a single port system with continuous time.

Another study that considered demand uncertainty is the work of Song and Dong (2008). They developed an ECR threshold control strategy under an uncertain demand of cyclic service routes. They formulated a mathematical model to obtain the optimal empty container repositioning policy in a dynamic and stochastic environment. Later, Zhang et al. (2014) extended the threshold control strategy to a multi-port system. Dong and Song (2009) also studied ECR management under uncertainty. They investigated fleet sizing in addition to empty container repositioning in a system with uncertain demand. Finally, studies by Song and Dong (2011), Song and Dong (2013) and Wang (2013), also focused on ECR management in addition to cargo routing problems with uncertain demand and multiple service routes.

Considering the above-mentioned studies in uncertain contexts, one can observe the application of
different management science techniques to handle empty container repositioning in a network. Some of these techniques include the simulation model (Lai et al., 1995; Song & Dong, 2008; Dong & Song, 2009; Song & Dong, 2011; Zhang et al., 2014), the integer programming and network model (Cheung & Chen, 1998; Song & Dong, 2013; Wang, 2013), the dynamic programming model (Crainic et al., 1993; Li et al., 2004; Song & Earl, 2008; Lam et al., 2007; Song & Zhang, 2010), and the inventory-based heuristic policy (Li et al., 2007).

As suggested in the literature, pricing can be an effective method to manage ECR and maximize earned profits. Gorman (2001) is one of the initial studies that used this approach as a strategy in ECR management. The author developed a methodology to maximize gained profits in a transportation system, considering both the market conditions and the empty equipment repositioning. In another study, Gorman (2002) developed a simulation model to improve the profitability of a freight transportation provider by reducing the equipment repositioning costs in an uncertain market. Zhou and Lee (2009) and Chen et al. (2016) are other studies that have investigated the pricing and revenue management problem in a maritime transportation market with two competing firms and deterministic demand. Zhou and Lee (2009) analyzed each firm’s profit function to make a decision about the price of the transportation service between two ports, and Chen et al. (2016) used these analyses to determine the price of the transportation service for both products and the associated amount of waste transported between the two ports. Although these studies utilized a pricing approach to maximize earned profits, they did not consider two important issues: the impact of service quality on price and demand uncertainty.

Yu et al. (2019) extended the work of Chen et al. (2016) to a heterogeneous system. They studied the pricing competition in a maritime transportation system that involved two heterogeneous ocean carriers providing transportation services between two ports. They investigated the issue of empty container repositioning by formulating the problem as a pricing game and deriving the price equilibrium for the two ocean carriers under both homogeneous and heterogeneous conditions. Zhang and Zhang (2020) is another recent study that investigated the impact of ECR in pricing. They developed a mathematical model to examine how freight forwarders determine pricing decisions with and without ECR cost sharing. They
found that the empty equipment balancing and ECR costs are some of the determinants of the pricing decisions. Further information about the pricing and revenue management for container liner shipping services can be found in the comprehensive work of Meng et al. (2019).

Topaloglu and Powell (2007) is one of the first studies that captured demand uncertainty in transportation service pricing. They investigated a transportation system with a homogeneous fleet of vehicles and uncertain demands, and made decisions for the movement of empty equipment. They developed a multi-period model to set the price at the beginning of each period and manage the fleet according to the set price in that period. Zheng et al. (2017) is another research study that used pricing strategy in maritime transportation services. They discussed the effects of risk-aversion on competing shipping lines’ pricing strategies with uncertain demands. To this end, they considered two carriers, where the first carrier was risk-neutral with sufficient capacity, and the second carrier was risk-averse with limited capacity. They used conditional value at risk (CVaR) to measure the risk-averse attitude of the second carrier, and derived a threshold for the optimal prices regarding the capacity of the second carrier. Despite this study investigating demand uncertainty in pricing, they did not develop a closed-form solution for the optimal price due to the problem complexity. In addition, similar to previous studies, it did not take into account the impact of service quality on the realized demand, and consequently, on the earned profit. Although different aspects of service quality, and their relation to the profits of carriers, have been considered in the literature on container transportation systems (e.g. Saura et al., 2008; Gang, 2013; Yuen and Thai, 2015; Han et al., 2017), to the best of our knowledge, none of these studies have taken into account pricing issues and empty container repositioning. Given the limitations discussed, the current study offers the following contributions to fill the existing gaps in the literature:

- This study develops a pricing and quality setting strategy for a maritime transportation service provider with a limited number of containers that need to be repositioned in the case of imbalanced demand and supply. The developed strategy also considers the impact of service quality on the realized demand and captures its effects on the price and earned profit.
• Considering demand uncertainty in the developed strategy is another contribution of our study. Regarding the uncertain nature of potential demand, the current study utilizes a robust optimization technique to develop a robust pricing and quality setting strategy. This strategy determines the service prices, considering different possible states of demand materialization, and results in efficient outcomes for the firm. In addition, this study presents an algorithm, according to the developed strategy, that assists the firm with choosing the optimal prices and service quality level.

• The current research also investigates the impact of different parameters on the firm’s earned profit. The results obtained by this analysis, presented as remarks, provide practical insights for the firm and allow for it to make more informed decisions.

3. Problem Definition

Consider a firm providing transportation service between two nodes: \( A \) and \( B \). There are two directions between these two nodes (i.e., AB and BA) and each one has its own potential demand. For the sake of simplicity, we have shortened direction AB to “direction a”, and direction BA to “direction b”. Load movements (demand) between these two nodes should be carried out by containers and there are a limited number of containers for moving these loads. Due to the limited amount of containers, the empty containers cannot be amassed at the destination node for a long time and should be returned to their origin for future shipments. Consequently, the firm may need to reposition the empty containers from one node to another in the case of imbalanced demand in order to ensure that further demand fulfillment is possible. The firm should set the price for these directions. Since repositioning imposes considerable costs on the shipping companies, and results in environmental and sustainability impacts such as fuel consumption and emissions, ECR is one of the main practical problems of transportation companies (Yu et al., 2019; Zheng et al., 2017; Zheng et al., 2015; Song and Carter, 2009; Zhou and Lee, 2009).

Setting quality for service is another concern for the firm. Several dimensions characterize the quality of service for the firm, including on-time pick-up and delivery, transit speed, and the loss and damage rate (Thai, 2008). Despite the fact that these factors can create many combinations for service quality, we only
focus on two levels of service quality to analytically investigate the problem under consideration. More precisely, the firm can either provide basic (regular) services or use different practices (e.g., using new IT and technological solutions) to improve the aforementioned quality factors and offer additional value-added services to their clients. Although these value-added services enhance the quality of service (Thai, 2007; Thai et al., 2014) and thereby increase the potential demand of the market (Yuen and Thai, 2015), they magnify shipping firms’ service costs. Regarding this twofold effect, shipping firms have to decide whether or not offer value-added services considering the total impact on the earned profit. To answer this question, consistent with the literature (e.g., Gang, 2013), this study investigates an abstracted form of service quality diversities to obtain a closed-form solution. That is, it considers the service quality variation as a binary variable to indicate whether the shipping firms present additional value-added service or not. Hence, we classify shipping services into two categories: (1) regular shipping (RS), and (2) high-quality shipping (QS). The former is common in the market and the latter offers some value-added services beyond the RS services that increase customers’ satisfaction.

In addition, regardless of the chosen level of service, both directions \(a\) and \(b\) should have the same quality of service (either QS or RS). Furthermore, the firm knows that the realized demand in each direction not only depends on the corresponding potential demand but also on the price set for that direction and the quality of service provided by the firm. Therefore, even if the QS service increases the potential demands in both directions \(a\) and \(b\) (Gang, 2013; Han et al., 2017) equally, its final impact on the realized demands will not necessarily be the same. More precisely, the impact of high-quality service on the realized demands depends on both its effect on the potential demands and the price set in those directions. Due to these twofold effects, service quality could indirectly influence the empty container repositioning amount, making it larger, lower, or unchanged, and should be jointly considered in the problem under investigation.

Regarding the issues mentioned, the current study aims to develop a Price and Quality Setting (PQS) strategy for maritime transportation services between two regions/nodes. Note that although the quality of service is the same in both directions \(a\) and \(b\), the price of service is not necessarily identical. Furthermore, consistent with reality, it is fair to assume that the price of transportation service in one direction does not
affect the amount of demand realized in the opposite direction. In spite of this independency, all parameters of both directions should be taken into account to determine the value and direction of empty repositioning. Therefore, the firm should investigate all parameters simultaneously in order to determine the value of decision variables (i.e., the quality level and the price of each direction).

We also investigate the impact of uncertainty on the pricing and quality level, and discuss how the firm can adopt an appropriate PQS strategy when the potential demand is uncertain. To this end, similar to literature (e.g. Zheng et al., 2017), we assume that the potential demand in each direction can vary within a range. In addition, we presume that all of the other parameters of the demand functions are deterministic and common knowledge for the firm. Moreover, we consider two types of cost: loaded movement and empty repositioning, which are incurred when providing transportation service. The former is incurred when the loads are transported, and it is influenced by the quality of service such that the firm incurs a larger transportation cost in the QS level of service quality. The latter is incurred when the empty containers are repositioned between two nodes. This type of cost is independent of the service quality, and it is often less than the loaded movement cost. To analyze the problem under investigation, the main notations including the parameters and decision variables are introduced below. For a more clear presentation, we gradually introduce the rest of the notations throughout the paper when necessary. The complete list of notations is also provided in Appendix A.

- **Notations**

  - \( s \) : Indicates a specific direction between two nodes, \( s \in \{a,b\} \),
  
  - \( D' \) : \textbf{Potential} demand volume in direction \( s \) in the deterministic case,
  
  - \( d' \) : \textbf{Realized} demand volume in direction \( s \) in the deterministic case,
  
  - \( \alpha' \) : The price sensitivity, measuring the demand responsiveness to the price in direction \( s \),
  
  - \( \beta' \) : The quality sensitivity, measuring the demand responsiveness to the level of service quality in direction \( s \),
  
  - \( c' \) : The loaded movement cost in direction \( s \),
\( \omega' \): The amount of growth in the loaded movement cost of direction \( s \) if the firm chooses the higher level of service quality.

\( \xi' \): The empty repositioning cost in direction \( s \).

\( p' \): The unit price of transportation service in direction \( s \)

\( q \): 1 if the higher level of quality is chosen for transportation service; 0 otherwise.

Although this research investigates potential demand uncertainty, for the sake of simplicity we first assume all the parameters of the demand functions are deterministic and common knowledge for the firm. We then consider the problem under uncertainty, and discuss how the firm should make decisions (i.e., determining the service quality level and setting prices) in the case of uncertain demand. In both cases, there are three potential policies to determine the service prices and to set the quality level according to the conditions of potential imbalance. The first policy, called Policy \( P_1 \), defines the service prices and the quality so that empty containers are repositioned in direction \( b \) (i.e., from \( B \) to \( A \)). This policy is adopted when the potential demand in direction \( a \) is significantly larger than the potential demand in the opposite direction (this condition is called the \( S_1 \) state). Conversely, the second policy, named Policy \( P_2 \), acts against Policy \( P_1 \) and sets the prices and the quality level in a way that the empty container repositioning occurs in direction \( a \) (i.e., from \( A \) to \( B \)). This policy is intuitively chosen when the potential demand in direction \( a \) is considerably less than the amount of potential demand in direction \( b \) (this condition is called the \( S_2 \) state).

Finally, the last policy, Policy \( P_3 \), aims to balance the realized demand in both directions. In other words, this policy defines the service prices and quality so that the firm will not incur any empty repositioning. This policy is adopted in conditions where the potential demand in both \( a \) and \( b \) directions are close enough (named the \( S_3 \) state). Naturally, these policies result in different prices, quality levels, and consequently, lead to different profits for the firm. Choosing the best policy and determining the optimal prices and level of service quality considering all the network parameters is called the Pricing and Quality Setting (PQS) strategy.

**4. Pricing and Quality Setting (PQS) Strategy in a Deterministic System**

To adopt a PQS strategy in the presence of empty repositioning and options for service quality, we first
assume that all the parameters of the demand function are deterministic. Moreover, it is presumed that the realized demand in each direction depends linearly upon the corresponding price and the quality of service. Hence, as discussed in the literature (Huang et al., 2013), the realized demand functions are as follows:

\[ d_a^a = D^a - \alpha^a p^a + \beta^a q \]  \hspace{1cm} (1)

\[ d_b^b = D^b - \alpha^b p^b + \beta^b q \]  \hspace{1cm} (2)

where \( D^a \) and \( D^b \) respectively show deterministic values of the potential demand in directions \( a \) and \( b \). Furthermore, \( d_a^a \) and \( d_b^b \) represent deterministic values of the realized demand in directions \( a \) and \( b \) respectively. In addition, the firm aims to define the service prices and quality in order to maximize the following profit function:

\[ II = (p^e - c^e - \omega^e q)d^e + (p^b - c^b - \omega^b q)d^b - \xi^e (d^e - d^e)^+ - \xi^b (d^b - d^b)^+ \]  \hspace{1cm} (3)

where \( II \) indicates the profit gained by the company throughout the planning horizon and \((x)^+ = \max \{0, x\}\). Given the three possible policies and defined states, the following two propositions summarize the firm’s PQS strategy when all the parameters are deterministic.

**Proposition 1.** The optimal pricing strategy of the firm is obtained with Equation (4) given that all the parameters are known with certainty.

\[
\begin{align*}
(p^*, p^*) &= \begin{cases}
  \left( \frac{D^e + \beta^e q + c^e + \omega^e q + \xi^b}{2\alpha^e} + \frac{D^b + \beta^b q + c^b + \omega^b q - \xi^b}{2} \right) & \text{if } \Lambda D \in (T_{j}^l, \infty) = S_1 \quad \text{(a)} \\
  \left( \frac{D^e + \beta^e q + c^e + \omega^e q - \xi^b}{2\alpha^e} + \frac{D^b + \beta^b q + c^b + \omega^b q + \xi^b}{2} \right) & \text{if } \Lambda D \in (-\infty, T_{j}^l) = S_2 \quad \text{(b)} \\
  \left( \frac{D^e + \beta^e q - (D^e + \beta^e q) + \alpha^e (c^e + \omega^e q + c^b + \omega^b q)}{2\alpha^e} \right) & \text{if } \Lambda D \in [T_{j}^l, T_{j}^u] = S_3 \quad \text{(c)}
\end{cases}
\end{align*}
\]

where \( T_{j}^l = \alpha^e (c^e + \omega^e q + \xi^b) - \alpha^b (c^b + \omega^b q - \xi^b) - (\beta^e - \beta^b)q \), \( T_{j}^u = \alpha^e (c^e + \omega^e q - \xi^b) - \alpha^b (c^b + \omega^b q + \xi^b) - (\beta^e - \beta^b)q \) and \( \Lambda D = D^e - D^b \).

**Proof.** The proof of Proposition 1 is presented in Appendix B.

As discussed in Appendix B, three distinct policies have been used in three possible states (\( S_1, S_2 \), and \( S_3 \)) of potential imbalance. Simply put, we derived several thresholds to guide us on when to use each
policy. The first policy, Policy $P_1$, is adopted in state $S_1$ where the potential imbalance ($\Delta D$) is larger than a predefined upper limit (i.e., $T^U_d$). In contrast, the second policy, Policy $P_2$, is chosen when $\Delta D$ is smaller than a predefined lower limit (i.e., $T^L_d$). Finally, the last policy, Policy $P_3$, is selected to balance the realized demand in both directions when the potential demands in both $a$ and $b$ directions are close enough (i.e., $\Delta D \in [T^L_d, T^U_d]$).

Considering the discussion of pricing policies, the most important parameters affecting the service price are the potential demands. These parameters not only determine the appropriate policy (either $P_1$, $P_2$, or $P_3$), but also affect the value of the service price and the other corresponding outcomes such as the realized demands and the profit, as depicted in Table 1. As shown in Equation (4) and Table 1, the service quality variable is another significant factor for pricing values and outcomes. Consequently, the service quality level should be set before determining the price of service in directions $a$ and $b$. Proposition 2 determines the optimal value of the service quality considering the values of potential demands.

### Table 1. Optimal profit and corresponding outcomes in the deterministic case

<table>
<thead>
<tr>
<th>$\Delta D$</th>
<th>$d^{a*}$</th>
<th>$d^{b*}$</th>
<th>$\Pi^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$\frac{1}{2}(D' + \beta' q - \alpha'(c' + \alpha' q + \xi'))$</td>
<td>$\frac{1}{2}(D' + \beta' q - \alpha'(c' + \alpha' q - \xi'))$</td>
<td>$\frac{(D' + \beta' q - \alpha'(c' + \alpha' q + \xi'))^2}{4\alpha^2}$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$\frac{1}{2}(D' + \beta' q - \alpha'(c' + \alpha' q - \xi'))$</td>
<td>$\frac{1}{2}(D' + \beta' q - \alpha'(c' + \alpha' q + \xi'))$</td>
<td>$\frac{(D' + \beta' q - \alpha'(c' + \alpha' q - \xi'))^2}{4\alpha^2}$</td>
</tr>
<tr>
<td>$S_i$</td>
<td>$\frac{\alpha'(D' + \beta' q) + \alpha'(D' + \beta' q - \alpha'(c' + \alpha' q + \xi'))}{2(\alpha' + \alpha'^*)}$</td>
<td>$\frac{\alpha'(D' + \beta' q) + \alpha'(D' + \beta' q - \alpha'(c' + \alpha' q + \xi'))}{4\alpha^2(\alpha' + \alpha'^*)}$</td>
<td></td>
</tr>
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</table>

**Proposition 2.** The optimal service quality level is as follows:

$$q^* = \begin{cases} 
1 & (\nu^a \geq 0 \text{ and } \nu^b \geq 0) \text{ or } (\nu^a \nu^b < 0 \text{ and } \Omega_q^{Lz} > 0) \\
0 & (\nu^a < 0 \text{ and } \nu^b < 0) \text{ or } (\nu^a \nu^b < 0 \text{ and } \Omega_q^{Lz} < 0)
\end{cases}$$

(5)

where $q = 1$ indicates QS service and $q = 0$ refers to RS service. Furthermore, $\nu^a = \beta^a - \alpha^a \omega^a$ and Equations (6) and (7) are used to calculate $\Omega_q^{Lz}$ and $\Omega_q^{Rz}$, respectively.

$$\Omega_q^{Lz} = \frac{2\kappa_a^a \nu^a + (\nu^a)^2}{4\alpha^a} + \frac{2\kappa_b^a \nu^b + (\nu^b)^2}{4\alpha^b}$$

$$\kappa_a^a = D^a - \alpha^a \left( c^a - \left(1 - \theta^a\right) \xi^a + \theta^a \xi^b \right)$$

$$\kappa_b^a = D^b - \alpha^a \left( c^b + \theta^a \xi^b - \left(1 - \theta^a\right) \xi^b \right)$$

(6)

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\[ \Omega_{q}^{U, a} = \frac{2\kappa_{q}^{a} v^{a} + (v^{a})^2}{4\alpha^{a}} + \frac{2\kappa_{q}^{b} v^{b} + (v^{b})^2}{4\alpha^{b}} \]
\[ \kappa_{q}^{a} = D^a - \alpha^a \left( c^a - \theta^a z^a + (1 - \theta^a) z^b \right) \]
\[ \kappa_{q}^{b} = D^b - \alpha^a \left( c^b + (1 - \theta^a) z^b - \theta^a z^b \right) \]

(7)

Where \( \theta^a \) equals 1 if \( \nu^a \) is positive, and 0 if \( \nu^a \) is negative.

**Proof.** The proof of Proposition 2 is presented in Appendix C.

Note that improving the service quality has a twofold effect on the firm’s profit. On one hand, it increases the cost of service (i.e., the loaded movement cost), and consequently, reduces the realized demand (i.e., \(-\alpha' w'\)). On the other hand, it enhances the realized demand by \( \beta' \) amount due to the higher quality. Hence, \( \nu' \) represents the quality improvement effect where the positive value of \( \nu' \) indicates that the positive impact on the realized demand outweighs its negative effect on the cost of service. Conversely, the negative value of \( \nu' \) indicates that the negative impact of a higher service quality is bigger than its positive impact. As Equation (5) demonstrates, the optimal value of \( q \) can easily be obtained if one level of service quality results in a higher profit in both directions \( a \) and \( b \). Otherwise, the decision is more complex when there is a mixed impact, i.e., one quality level (either QS or RS) results in a higher profit in one direction while the profit deteriorates in another direction. Therefore, we have to take into account the impact of service quality in both directions \( a \) and \( b \) in order to choose the optimal service quality level.

Considering the complexity of related equations (See Appendix C), a closed-form solution could not be developed to compute the change in total profit based on different service quality levels. However, we are able to develop two thresholds that can assist us in selecting the optimal service quality level in the case of mixed impact. \( \Omega_{q}^{L, a} \) indicates a lower bound and \( \Omega_{q}^{U, a} \) denotes an upper bound of change in total profit as a result of selecting a high service quality (QS). Therefore, if \( \Omega_{q}^{L, a} \) (\( \Omega_{q}^{U, a} \)) is not negative (positive), it is still recommended to offer a high (regular) service quality. Finally, it is worth mentioning that Proposition 2 determines the optimal quality level for most cases. However, there are some cases in which none of the conditions of Equation (5) are satisfied. In these circumstances, the optimal value of the quality variable should be determined numerically.

Given the results obtained with Propositions 1 & 2, and our discussion above, the steps for determining
the optimal PQS strategy are shown in Fig. 1. As this figure depicts, the values of \( v_a \) and \( v_b \) are first calculated. These values often determine the optimal level of service quality for the firm according to the conditions developed by Proposition 2. If the conditions are enough to find the optimal service quality level, the service prices are calculated using Equation (4) as depicted in Box A of Fig. 1. Otherwise, thresholds \( \Omega_{q}^{Lq} \) and \( \Omega_{q}^{Uq} \) are calculated to determine the optimal level of service quality. If using these thresholds cannot determine the optimal value of \( q \), then the optimal service quality is determined numerically (Box B of Fig. 1). To this end, the optimal prices and the corresponding profits are calculated for each quality level (i.e., \( q = 0 \) and \( q = 1 \)) and the level with the largest profit is selected.

5. Pricing and Quality Setting (PQS) Strategy in an Uncertain Environment

In this section, we investigate the PQS strategy for transportation service in the presence of uncertain potential demand. Similar to Zheng et al. (2017), we assume that \( \hat{D}^x \) and \( \hat{D}^y \) are uncertain and change uniformly in the ranges of \([D_1^x, D_m^x]\) and \([D_1^y, D_m^y]\), respectively. In order to determine the PQS strategy under uncertainty, additional notations are defined as follows:

- \( D_1^s, D_m^s \): The lower and upper levels of potential demand in direction \( s \), respectively.
- \( \hat{D}^s \): A uniform random variable showing the potential demand volume in direction \( s \), \( \hat{D}^s \in [D_1^s, D_m^s] \).
- \( \hat{\Omega}^s \): The materialized value of potential demand in direction \( s \),
- \( \hat{d}^s \): The volume of realized demand in direction \( s \) under uncertainty.

To handle this type of uncertainty, we use a robust optimization technique developed by Aghezzaf et al. (2010). This technique investigates both expected and pessimistic outcomes and enables the user to create a balance between them according to his/her risk preferences. More precisely, this method is an extension of the proposed approach by Mulvey et al. (1995) that incorporated both the maximum value and the
Start

Calculate \( \nu^a \) and \( \nu^b \)

If \( \nu^a \), \( \nu^b < 0 \)

Investigate the obtained values

If \( \nu^a \geq 0 \) and \( \nu^b \geq 0 \)

Set \( q^* = 1 \)

If \( \nu^a < 0 \) and \( \nu^b < 0 \)

Set \( q^* = 0 \)

Calculate \( \theta^a \) and \( \theta^b \)

Compute \( Q^a \) and \( Q^b \)

If \( Q^a \geq 0 \)

Set \( q^* = 1 \)

If \( Q^a \leq 0 \)

Set \( q^* = 0 \)

A: Calculate the Price

Calculate \( T^u \) and \( T^l \)

If \( \Lambda_D > T^u \)

Calculate the price by Eq. (4.a)

Investigate \( \Lambda_D \in [T^l, T^u] \)

Calculate the price by Eq. (4.b)

If \( \Lambda_D < T^l \)

Calculate the price by Eq. (4.c)

B: Compute \( q^* \) Numerically

Set \( q = 0 \)

If \( q \leq 1 \)

No

Yes

Set \( q = q + 1 \)

Read the prices \((P^a, P^b)\)

Compute \( H_q \)

Compare \( H_q \) to \( H_q \)

If \( H_q \leq H_q \)

Set \( q^* = 1 \) and \( H_q = H_q \)

Set \( q^* = 0 \) and \( H_q = H_q \)

End

Fig. 1. Proposed algorithm for PQS strategy in the case of deterministic parameters
average amount of regret as representations of the pessimistic and expected outcomes respectively. In other words, as Equation (8) shows, this approach minimizes the combination of the maximum anticipated regret \( \max(\mathcal{R}) \) and the expected anticipated regret \( E(\mathcal{R}) \) using a control factor called the pessimism control parameter, referred to as the control parameter for short, denoted with \( \lambda \). For simplicity, we denote variable \( \mathcal{R} \) with EM-Reg, and use the term regret throughout the rest of this paper when referring to anticipated regret.

\[
\min \mathcal{R} = \lambda E(\mathcal{R}) + (1-\lambda)\max(\mathcal{R})
\] (8)

Given this equation, a larger value of control parameter \( \lambda \) limits the decision maker from considering the worst-case situations where the maximum regret occurs. Therefore, \( \lambda = 1 \) disregards the maximum regret and only considers the expected regret. It is evident that EM-Reg (i.e., \( \mathcal{R} \)) with a smaller value of \( \lambda \) is more appropriate for decision makers who are more concerned about larger regret amounts.

As mentioned in Section 4, the potential demand parameter is an important factor and has a twofold effect on the PQS strategy. On one side, their difference (i.e., potential demand imbalance) determines the appropriate policy among the available policies \( P_1, P_2 \) and \( P_3 \) considering the occurred state of potential imbalance (\( S_1, S_2 \) or \( S_3 \)). On the other side, its value affects the service price of a selected policy. To investigate the impact of uncertainty, we first consider its impact on the pricing values as shown in Proposition 3 and Corollary 1. Then, we discuss the probability of occurrence for each state (\( S_1, S_2 \) or \( S_3 \)) through Proposition 4.

**Proposition 3.** Given a specific \( \lambda \) value, considering \( \mathcal{D}^s = D^s + \frac{\lambda}{2}(D^s - D^s_\hat{s}) \) and \( \mathcal{D}^b = D^b + \frac{\lambda}{2}(D^b - D^b_\hat{s}) \) as the materialized values of \( \mathcal{D}^s \) and \( \mathcal{D}^b \) in determining service prices minimizes EM-Reg (i.e., \( \mathcal{R} \)).

**Proof.** The proof of Proposition 3 is presented in Appendix D.

Note that \( \mathcal{D}^s (s \in \{a, b\}) \) is a point estimation for the materialized value of potential demand \( \mathcal{D}^s \) (i.e., \( \mathcal{D}^s \)) in the PQS strategy and varies from the average potential demand to the upper limit of potential demand considering the control parameter (i.e. \( \lambda \)).
**Corollary 1.** Pricing the transportation services based on the average potential demands (i.e., $E(\tilde{D}^r)$ and $E(\tilde{D}^b)$) minimizes the expected regret.

**Proof.** The proof of **Corollary 1** is expressed in Appendix D.

According to this corollary, if the decision maker would like to minimize the expected value of regret by disregarding its maximum value, he/she could simply set the materialized value of potential demand to the average values.

It is evident that the potential imbalance ($\Delta D$) is uncertain because of potential demand uncertainty in directions $a$ and $b$ (i.e., $\tilde{D}^r$ and $\tilde{D}^b$). Therefore, it is possible that each state $S_1$, $S_2$, and $S_3$ arises as a result of potential demand materialization. Thus, we compute the corresponding probabilities with respect to the ranges of potential demands (i.e., $[D^r_n, D^b_n]$ and $[D^b_n, D^b_m]$). Given that these probabilities are $P(\Delta D \in S_1) = p_1$, $P(\Delta D \in S_2) = p_2$, and $P(\Delta D \in S_3) = p_3$, **Proposition 4** calculates them based on the ranges of potential demands.

**Proposition 4.** Considering the variation ranges of $\tilde{D}^r$ and $\tilde{D}^b$, $L = T_i^j$ and $T = T_i^j$, the values of $p_1$, $p_2$, and $p_3$ are computed as follows:

\[
p_i = \begin{cases} 
0 & D^r_n - D^b_n \leq T \\
\frac{(D^r_n - D^b_n - T)^2}{2R^2R^2} & D^r_n - D^b_n \leq T \land D^r_n - D^b_n \leq T \\
\frac{2(D^r_n - T) - (D^r_n + D^b_n)}{2R^2} & D^r_n - D^b_n \leq T \land D^r_n - D^b_n \geq T \\
\frac{(D^r_n + D^b_n) - 2(D^r_n + T)}{2R^2} & D^r_n - D^b_n \geq T \land D^r_n - D^b_n \leq T \\
1 & D^r_n - D^b_n \geq T \land D^r_n - D^b_n \geq T \\
1 - \frac{(D^b_n - D^b_n + T)^2}{2R^2R^2} & D^r_n - D^b_n \geq T \\
1 & D^r_n - D^b_n \geq T 
\end{cases}
\]  

(9)
\[
\begin{align*}
\begin{cases}
0 & \text{if } D^a_n - D^b_n \geq L \\
\left(\frac{(D^a_n - D^b_n + L)^2}{2R_n^a R_n^b}\right) & \text{if } D^a_n - D^b_n \geq L \land D^e_n - D^b_n \geq L \\
2\left(D^a_n + L\right) - \left(D^e_n + D^b_n\right) & \text{if } D^a_n - D^b_n \geq L \land D^e_n - D^b_n \leq L \\
\left(\frac{(D^e_n + D^b_n) - 2(D^a_n - L)}{2R_n^e}\right) & \text{if } D^a_n - D^b_n \leq L \land D^e_n - D^b_n \geq L \\
1 - \left(\frac{(D^e_n - D^b_n - L)^2}{2R_n^e R_n^b}\right) & \text{if } D^a_n - D^b_n \leq L \land D^e_n - D^b_n \leq L \\
1 & \text{if } D^a_n - D^b_n \leq L \\
\end{cases}
\end{align*}
\]

\[
p_2 = \frac{\left(\frac{(D^a_n - D^b_n + L)^2}{2R_n^a R_n^b}\right)}{2R_n^a R_n^b} \\
p_3 = 1 - p_1 - p_2
\]

**Proof.** The proof of Proposition 4 is presented in Appendix E.

Considering the obtained results in this proposition, all three states \(S_1, S_2,\) and \(S_3\) may occur after the materialization of potential demands. Therefore, all the policies \(P_1, P_2\) and \(P_3\) could potentially be the most appropriate policy for pricing. In this case, the best strategy is that the firm should investigate all possible policies and their corresponding regrets. The policy that leads to the least regret should then be selected as the most appropriate policy. In doing so, Propositions (5) to (7) respectively calculate the regret for policies \(P_1, P_2\) and \(P_3\) given that each state \(S_1, S_2,\) or \(S_3\) can occur.

**Proposition 5.** Adopting Policy \(P_1\) will respectively result in the following regrets given each state of potential imbalance.

\[
\begin{align*}
\text{Reg}(P_1 | S_1) &= \left(\frac{\Delta^a}{4a^e}\right)^2 + \left(\frac{\Delta^b}{4a^b}\right)^2 \\
\text{Reg}(P_1 | S_2) &= \left(\frac{\Delta^a - \alpha^a \left(\xi^a + \xi^b\right)}{4a^e}\right)^2 + \left(\frac{\Delta^a + \alpha^e \left(\xi^a + \xi^b\right)}{4a^b}\right)^2 \\
\text{Reg}(P_1 | S_3) &= \left(\frac{\Delta^a}{4a^e}\right)^2 + \left(\frac{\Delta^b}{4a^b}\right)^2 + \left(\frac{\Delta^b}{4a^b}\right)^2 + \left(\frac{\Delta^b}{4a^b}\right)^2 - \left(\frac{\Delta^b}{4a^b}\right)^2 \\
\end{align*}
\]

where,

\(\hat{D}^a:\) The materialized value of potential demand in direction \(a,\)

\(\hat{D}^b:\) The materialized value of potential demand in direction \(b,\)

\(\Lambda\hat{D}:\) The materialized value of the potential imbalance,
\( \Delta^a \): Difference between \( \hat{D}^a \) and \( D^a \) (i.e. \( \hat{D}^a - D^a \)),

\( \Delta^b \): Difference between \( \hat{D}^b \) and \( D^b \) (i.e. \( \hat{D}^b - D^b \)).

**Proof.** The detailed proof of **Proposition 5** is presented in **Appendix F**.

**Proposition 6.** Adopting Policy \( P_2 \) will respectively generate the following regrets if the state of potential imbalance \( (S_1, S_2, \text{ or } S_3) \) is given.

\[
\begin{align*}
\text{Reg}(P_2 | S_1) &= \frac{\left( \Delta^a + \alpha^a \left( \xi^a + \xi^b \right) \right)^2}{4\alpha^a} + \frac{\left( \Delta^b + \alpha^b \left( \xi^a + \xi^b \right) \right)^2}{4\alpha^b} \quad \Lambda \hat{D} \in S_1 \\
\text{Reg}(P_2 | S_2) &= \frac{\Delta^a}{4\alpha^a} + \frac{\Delta^b}{4\alpha^b} \quad \Lambda \hat{D} \in S_2 \quad (13) \\
\text{Reg}(P_2 | S_3) &= \frac{\left( \Delta^a \right)^2}{4\alpha^a} + \frac{\left( \alpha^a \xi^a \right)^2}{4\alpha^a} + \frac{\left( \alpha^b \xi^b \right)^2}{4\alpha^b} + \frac{\left( \Delta^a - \Delta^b \right)^2}{2} \quad \Lambda \hat{D} \in S_3
\end{align*}
\]

**Proof.** The proof of **Proposition 6** is expressed in **Appendix G**.

**Proposition 7.** Adopting Policy \( P_3 \) will respectively lead to regret values given the state of potential imbalance \( S_1, S_2, \text{ or } S_3 \).

\[
\begin{align*}
\text{Reg}(P_3 | S_1) &= \frac{\left( \Delta^a - \alpha^a \xi^a \right)^2}{4\alpha^a} + \frac{\left( \Delta^b - \alpha^b \xi^a \right)^2}{4\alpha^b} \quad \Lambda \hat{D} \in S_1 \\
\text{Reg}(P_3 | S_2) &= \frac{\left( \Delta^a - \alpha^a \xi^a \right)^2}{4\alpha^a} + \frac{\left( \Delta^b + \alpha^b \xi^a \right)^2}{4\alpha^b} \quad \Lambda \hat{D} \in S_2 \quad (14) \\
\text{Reg}(P_3 | S_3) &= \frac{\left( \Delta^a \right)^2}{4\alpha^a} + \frac{\left( \Delta^b \right)^2}{4\alpha^b} + p_1^r \frac{\xi^b}{2} \left( \Delta^a - \Delta^b \right) + p_2^r \frac{\xi^a}{2} \left( \Delta^b - \Delta^a \right) \quad \Lambda \hat{D} \in S_3
\end{align*}
\]

where \( p_1^r = P(\Delta^a \geq \Delta^b) \) and \( p_2^r = 1 - p_1^r = P(\Delta^a \leq \Delta^b) \).

**Proof.** The detailed proof of **Proposition 7** is presented in **Appendix H**.

It is also worth noting that the value of \( p_i^r \) is calculated as follows:

\[
p_i^r = P(\Delta^a - \Delta^b \geq 0) = P\left( (\hat{D}^a - D^a) - (\hat{D}^b - D^b) \geq 0 \right) = P\left( (\hat{D}^a - D^a) \geq (\hat{D}^b - D^b) \right) \\
= P\left( (\hat{D}^a - \hat{D}^b) \geq \left( D^a + \frac{\lambda}{2} R^a - D^b - \frac{\lambda}{2} R^b \right) \right) = P\left( (\hat{D}^a - \hat{D}^b) \geq D^a + \frac{\lambda}{2} (R^a - R^b) \right) \quad (15)
\]

where \( R^a = D^a - D^a \) and \( R^b = D^b - D^b \). Now, applying **Proposition 4** concludes:
To choose the optimal pricing policy for transportation services in directions $a$ and $b$, the firm needs to calculate its EM-Reg value taking into account all potential imbalance states and their occurrence probabilities. It is rational for the firm to choose the optimal policy which results in the minimum value of expected EM-Reg. Proposition 8 shows how to obtain the optimal policy.

Proposition 8. The optimal pricing policy can be chosen as

$$P^* = \arg \min \ E^{\text{opt}} \left\{ P \mid P \in \mathcal{P} \ & \forall P' \in \mathcal{P} : E^{\text{opt}}(P) \leq E^{\text{opt}}(P') \right\}$$

where $\mathcal{P} = \{P_1, P_2, P_3\}$ and

$$E^{\text{opt}} = \begin{cases} 
    p_1 \left( \frac{\xi^+}{4} \right)^2 \left( (1-\lambda)(R^+ - R^+) + (\alpha^+ + \alpha^-)(\xi^+ + \xi^-) \right) + p_2 \left( \frac{\alpha^+ + \alpha^-}{4} \right)^2 \left( 1 - \lambda \right) \xi^+ (R^+ - R^-) & \text{for } P_1 \\
    p_2 \left( \frac{\xi^+}{4} \right)^2 \left( (1-\lambda)(R^+ - R^+) + (\alpha^+ + \alpha^-)(\xi^+ + \xi^-) \right) + p_3 \left( \frac{\alpha^+ + \alpha^-}{4} \right)^2 \left( 1 - \lambda \right) \xi^- (R^+ - R^-) & \text{for } P_2 \\
    (1-\lambda)(R^+ - R^-) \left( p_1 + p_2 q_1 \right) \xi^+ - (p_1 + p_2 q_2) \xi^- + (\alpha^+ + \alpha^-) \left( \frac{p_1 (\xi^+)^2 + p_2 (\xi^-)^2}{4} \right) & \text{for } P_3.
\end{cases}$$

Proof. The proof of Proposition 8 is expressed in Appendix I.
Step 1. Compute the occurrence probability for all three possible states $S_1$, $S_2$, and $S_3$ by applying the results of Proposition 4.

Step 2. Calculate $p'_1$ and $p'_2$ values using Equation (16).

Step 3. Determine the optimal pricing policy with the aid of Proposition 8.

Step 4. Calculate $\bar{D}'$ and $\bar{P}$ values considering the control parameter ($\lambda$) through the equations obtained with Proposition 3.

Step 5. Apply the algorithm depicted in Fig. 1 to determine the prices and the quality of transportation services in directions $a$ and $b$.

6. Discussions and Numerical Illustrations

To investigate the behavior of the developed strategy and the impact of the parameters on the decisions made, this section analyzes the effects of different parameters on the firm’s regret and profit. This analysis encompasses all the main parameters, including: control parameter ($\lambda$), potential imbalance ($\Lambda$), empty repositioning cost ($\xi^r$), and price sensitivity ($\alpha^r$). In addition, to investigate the impact of uncertainty on the decisions, we assume that the potential demand belongs to a range $\tilde{D}' \in [\mu' - \kappa \mu', \mu' + \kappa \mu']$ with a variation coefficient ($\kappa$). Regarding the possible conditions of the parameters under investigation, we examine each parameter at three levels. More concisely, the control parameter (i.e., $\lambda$) is investigated at three levels 0.2, 0.5, and 0.8 being representative of circumstances in which the firm has attitude the maximum regret, no-attitude, and attitude the expected regret, respectively. The potential imbalance (i.e., $\Lambda$) is also considered at three levels $\Lambda > 0$, $\Lambda = 0$, and $\Lambda < 0$. The first level represents situations in which the average potential demand in direction $a$ is larger than the average potential demand in direction $b$. The second level stands for circumstances in which the average potential demands in both directions are the same. Lastly, the third level of $\Lambda$ is representative of cases in which the average potential demand in direction $a$ is less than average potential demand in direction $b$. To investigate the impact of the empty repositioning cost, we consider the ratio of this parameter to the loaded movement cost at three levels: 0.2, 0.5, and 0.8. Moreover, regarding the possible values of price sensitivity in directions $a$ and $b$, we categorize them into three cases, including $\alpha^a < \alpha^b$, $\alpha^a = \alpha^b$, and $\alpha^a > \alpha^b$. Finally, the variation coefficient is the last
parameter investigated at three levels 0.05, 0.15, and 0.25. Note that, these values respectively represent 10%, 30% and 50% variation of potential demands.

Given the number of factors at different levels, we use $3^5$ full factorial designs to examine whether different levels of these factors affect the firm’s regret or not. Therefore, we generate $3^5=243$ test cases regarding disparate levels of factors under investigation. For the sake of simplicity, each one of the 243 problems is called a Master Problem (MP). The PQSASS algorithm is used to determine service quality and prices in each MP. The obtained results are then utilized in a Monte Carlo Simulation analysis to analyze the impact of the parameters on the firm’s regret. To this end, 1000 instances are generated randomly based on the range of potential demands for each MP. Then, the prices and quality set by the PQSASS algorithm are applied in generated test instances to determine its associated regret. Finally, the average of the incurred regret in each MP is utilized for the statistical analysis.

As Table 2 shows, applying statistical analysis on the resulted regret reveals that three primary and two interaction effects have significant impacts on the firm’s regret. More precisely, the control parameter, the potential demand imbalance, and the variation coefficient have a significant impact on the firm’s regret. The results also show that the interaction of price sensitivity with the potential imbalance and the empty repositioning cost have a significant impact on the regret, although its main effect is not significant.

<table>
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<tr>
<th>Factor</th>
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<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
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<td>100.671</td>
<td>0.298</td>
</tr>
<tr>
<td>Price Sensitivity x Empty Repositioning Cost</td>
<td>$\alpha \times \xi$</td>
<td>94,233.57</td>
<td>4</td>
<td>23,558.392</td>
<td>69.769*</td>
</tr>
<tr>
<td>Price Sensitivity x Variation Coefficient</td>
<td>$\alpha \times \kappa$</td>
<td>1.19</td>
<td>4</td>
<td>0.297</td>
<td>0.001</td>
</tr>
<tr>
<td>Empty Repositioning Cost x Variation Coefficient</td>
<td>$\xi \times \kappa$</td>
<td>1,226.00</td>
<td>4</td>
<td>306.501</td>
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<tr>
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<td></td>
<td>64,832</td>
<td>192</td>
<td>337.665</td>
<td></td>
</tr>
</tbody>
</table>

Total: 15,605,014.69, 242

* $F_{0.95,2,192}=3.043; F_{0.95,4,192}=2.419.$
Regarding the effective parameters, we set another Monte Carlo Simulation analysis to investigate their impacts on the earned profit. To this end, all significant parameters except one are fixed and the parameter under investigation is changed in a predefined range. The generated problems are then solved using the PQSASS algorithm and their associated prices and quality level are calculated. Lastly, these decisions are analyzed using a Monte Carlo Simulation to compute the firm’s regret according to the decisions made. More precisely, we generate several problems, entitled Master Problems (MP), regarding the fixed and changing parameters in which the changing parameter is varied in the predetermined range (See Appendix A for details). Similar to the previous simulation analysis, 1000 instances are generated randomly based on the range of potential demands for each MP. The PQSASS algorithm is subsequently applied to determine the prices and the quality level in the generated instances. For the sake of simplicity, we call the obtained decisions of each generated Instance IPD (i.e. Instance Problem Decisions). Afterward, we compute the profits of IPD in all instances (called actual profit) and compare it against the optimal profit of each instance after the materialization of potential demands. Note that the optimal profit is the firm’s profit in the case of given potential demand and is computed by applying the materialized value of potential demands in the PQS algorithm. However, the actual profit is the profit earned in an uncertain circumstance and calculated by using the PQSASS algorithm. Since the PQSASS algorithm does not necessarily lead to an optimal solution for each instance, the actual profit is often less than the optimal profit. This difference creates some regret for the firm. Finally, to aggregate the obtained results, we calculate the average of the actual profits, optimal profits, and associated regrets. Please note that the number of instances is determined to ensure that the average regret lies within a 95% confidence interval.

Remark 1. In the case of uncertain potential demand, when the firm selects the control parameter \( \lambda \) closer to its extreme values (i.e. \( \lambda = 0 \) or \( \lambda = 1 \)), this decision does not necessarily improve its actual profit and can also lead to a larger regret value.

The results of the first remark can be explained by considering all three states: 1) \( \Lambda = \mu^c - \mu^b \in S_1 \), 2) \( \Lambda \in S_2 \), and 3) \( \Lambda \in S_3 \). As seen in Table 1, profit functions in all states \( S_1, S_2, \) and \( S_3 \) depend on the values
of $\bar{D}$. In addition, according to Proposition 3, the selected value of $\bar{D}$ is directly influenced by the control parameter $\lambda$. However, the realized values of the potential demands are independent of the control parameter. Therefore, the optimal profit does not change by altering the control parameter ($\lambda$), but the actual profit of the firm behaves differently because it is affected by both the optimal profit and the regret amount. Since $\lambda$ influences $\bar{D}$, adopting a low or high value of control parameter can keep $\bar{D}$ away from the materialized potential demand, and consequently, increases the regret amount.

**Fig. 2** illustrates the average value of the optimal profit and the corresponding actual profit (i.e. the profit obtained by applying the PQSASS algorithm) in one numerical illustration. As this figure demonstrates, the lower value of risk aversion does not necessarily translate to a larger profit. In addition, pricing transportation services based on the expected value of potential demand (i.e. $\lambda = 1$) does not lead to the largest profit. Furthermore, while setting $\lambda = 0.5$ in the EM-Reg function brings about the lowest regret and the highest value of actual profit (the dotted line), the pricing based on the expected value of potential demand (i.e. $\lambda = 1$) results in the lowest actual profit. Therefore, using the EM-Reg function rather than the expected regret function can improve the actual profit.

**Fig. 2.** Actual profit and regret variation against control parameter

($\mu^a = \{80000, 90000, 100000, 110000, 120000\}$, $\mu^b = \{120000, 110000, 100000, 90000, 80000\}$, $a^a=13$, $a^b=14$, $\kappa=0.1$, $\beta=0.1\mu^a$, $c^a=800$, $\omega^a=200$, $\xi^a=450$)
**Remark 2.** In the case of uncertain potential demand, if the total average potential demands (i.e. $\mu^a + \mu^b$) of the firm are fixed:

(a) The profit of the firm is increasing (decreasing) in potential imbalance (i.e., $\Lambda$) if $\alpha^a < \alpha^b$ ($\alpha^a > \alpha^b$).

(b) The profit of the firm is convex in potential imbalance (i.e., $\Lambda$) if $\alpha^a = \alpha^b$.

(c) The regret amount is decreasing in absolute value of potential imbalance (i.e., $|\Lambda|$).

The first part of this remark (Remark 2a) points out that a larger potential imbalance does not necessarily lower the profit. Since the potential imbalance is defined as $\Lambda = \mu^a - \mu^b$, and the average of the total market size (i.e., $\mu^a + \mu^b$) is assumed to be fixed, a larger potential demand in one direction means a lower potential demand in the opposite direction. To verify Remark 2a, it is enough to derive the profit functions of the states $S_1$, $S_2$, and $S_3$ with respect to $D^a$ and $D^b$. The derivative of the profit functions with respective to price sensitivity (i.e., $\alpha^a$ and $\alpha^b$) is obviously decreasing in both directions $a$ and $b$. In addition, due to the appearance of $\alpha^a$ and $\alpha^b$ parameters in the denominator of corresponding derivatives, the direction with the lower price sensitivity is more affected than the direction with the higher price sensitivity. Now assume that the potential imbalance increases by $k$ units. Based on the definition, the corresponding impact on the profit functions is respectively positive and negative in directions $a$ and $b$. If $\alpha^a < \alpha^b$, the positive profit enhancement in direction $a$ is more than the profit decrement in direction $b$. Consequently, the total impact on the profit is positive. On the contrary, if $\alpha^a > \alpha^b$, the profit increase in direction $a$ is less than the decrease in direction $b$. Thus, the total profit decreases due to potential imbalance.

To verify Remark 2b (i.e., the convex behavior of the profit function in potential imbalance for the case of the same sensitive market ($\alpha^a = \alpha^b$)), we need to check all the possible states. If $\Lambda \in S_1$, the corresponding profit function is independent from the potential imbalance because the profit increment in one direction equals the profit decrement in another direction. Hence, around $|\Lambda| = 0$, the total profit is unchanging in potential imbalance. Furthermore, if $\Lambda \in S_1$ or $S_2$, the partial derivatives of the corresponding profit function with respect to $D^a$ and $D^b$ are respectively $d_a^a/\alpha^a$ and $d_a^b/\alpha^b$. Since $\alpha^a = \alpha^b$, the change in the profit of
direction $a$ is more than the change in the profit of direction $b$ if $\Lambda \in S_1$. Hence, potential imbalance enhancement in this state brings about more profit for the firm. Contrarily, if $\Lambda \in S_2$, the amount of profit change in direction $a$ is less than the amount in direction $b$ because $d^a > d^b$. Therefore, any decrement in the potential imbalance reduces $d^a$, increases $d^b$, and consequently, enhances the total profit.

Finally, to verify Remark 2c, we use an intuitive outcome that the developed PQS strategy has the minimum regret if the realized state matches the state chosen by the strategy. For example, if policy $S_1$ is recommended, then it is observed that the materialized potential demands also create the $S_1$ state. This outcome can be also obtained from Propositions 5 to 7. Therefore, any change in parameter that assists the PQS strategy with selecting proper policy decreases the regret amount. Now, consider the potential imbalance parameter. A larger positive value of $\Lambda$ enhances the chance of state $S_1$ realization (that is $d^a > d^b$). At the same time, according to Proposition 4, increasing the probability of the $S_1$ state will incline the firm to select the first policy. Due to this compatibility, the regret amount decreases for the large value of $\Lambda$. Similarly, if $\Lambda$ becomes smaller while it is negative (i.e., its absolute value enhances), this change increases the chance of realization for state $S_2$. According to Proposition 4, the higher probability encourages the firm to select policy $P_2$, which consequently, reduces the regret amount. However, when the absolute value of potential imbalance decreases (especially when it changes around zero), it is intuitive that the chance of compatibility between what the model suggested as the optimal policy and the realized state will decrease, and result in a higher regret amount.

To illustrate these remarks, the results of a numerical example are shown in Fig. 3. As seen from this figure, a larger potential imbalance may result in a higher or lower amount of profit for the firm based the values of $\alpha$ (see Fig. 3a for $\alpha^a < \alpha^b$, Fig. 3b for $\alpha^a > \alpha^b$, and Fig. 3c for $\alpha^a = \alpha^b$). In addition, a larger positive or negative value of potential imbalance lowers the regret amount.
As mentioned earlier, to capture the potential demand uncertainty, it is presumed that the potential demand varies in a range following a uniform distribution (i.e., $D \in [D_M, D_A]$ where $D_M = \mu - \kappa \mu' \text{ and } D_A = \mu' + \kappa \mu'$). It is intuitive that $\kappa = 0$ means potential demands are deterministic and $\kappa = 0.5$ means potential demands have a large variation ($R' = \mu'$). **Remark 3** presents the impact of $\kappa$ on the anticipated regret.

**Remark 3.** In the case of uncertain potential demand, the firm’s regret is increasing in variation coefficient ($\kappa$).

Although it seems intuitive that increasing the potential demand variation enhances the regret amount, it can also be explained by the behavior of the PQS strategy. As mentioned in **Remark 2**, the developed PQS strategy results in less regret when it chooses the correct policy. It is evident that when the variation coefficient is zero (i.e. the potential demand is known with certainty), only one of the $S_1, S_2$ and $S_3$ states
can occur. In this case, the \textit{PQS} strategy selects the optimal policy. On the other hand, due to zero variability, the realized value of the potential demand equals the value considered in the \textit{PQS} strategy (i.e., $\Delta^a = \Delta^b = 0$). Hence, the proposed regret would be zero. Unlike the low variation, in which only one of the $S_1$, $S_2$ and $S_3$ states has a high chance of realization, at the high level of $\kappa$, all states have a relatively high probability of happening. For instance, if both directions $a$ and $b$ have the same parameters, the materialization probability of each state is 0.33 for $\kappa=0.5$. In this circumstance, there is a high chance (i.e., 0.66) that the policy selected by the \textit{PQS} strategy will not match the realized state, and therefore, it will lead to a large regret amount.

To better understand \textbf{Remark 3}, \textbf{Fig. 4} illustrates how the variation coefficient $\kappa$ can impact the optimal profit, the actual profit, and the regret amount based on a numerical example. As shown, a higher variation does not significantly affect the firm’s profit. However, it may slightly enhance the optimal profit because a high level of variation may bring about a higher potential demand. Nevertheless, the actual profit does not increase at the high levels of variation because of regret growth. As discussed, a growth in variation enhances the regret amount.

![Fig. 4. Actual profit and regret variation against variation coefficient](image)

\textbf{Remark 4.} In the case of uncertain potential demand,

(a) The profit of the firm is always decreasing in the unit loaded cost $C^a$ and $C^b$. 

(b) If $\Lambda \in S_i ( \in S_i \text{ or } S_i )$, the firm’s profit is unchanging (decreasing, unchanging) in empty repositioning cost $\xi^a$.

(c) If $\Lambda \in S_i ( \in S_i \text{ or } S_i )$, the firm’s profit is unchanging (decreasing, unchanging) in empty repositioning cost $\xi^b$.

It is common sense that the total profit decreases with a larger cost. In addition, the derivative of profit functions (previously presented in Table 1), with respect to the unit loaded cost parameters, verifies the validity of Remark 4a. The impact of the unit empty repositioning cost on the actual profit is also intuitive.

As mentioned in Proposition 1, if $\Lambda \in S_i$, the firm chooses policy $P_i$, and determines the prices so that $d^a > d^b$. In this case, the empty containers are repositioned in direction $b$. Hence, any change in the unit empty repositioning cost in direction $a$ (i.e., $\xi^a$) will not affect the total earned profit. However, if $\Lambda \in S_i$, the firm prefers policy $P_2$, and determines the prices so that $d^a < d^b$. Due to the empty repositioning in direction $a$, any increase of $\xi^a$ enhances the total empty repositioning cost, and consequently, reduces the total earned profit. Finally, if $\Lambda \in S_i$, policy $P_3$ outperforms the other policies. This policy determines the prices so that no empty repositioning is required. Hence, there is no dependency between the total earned profit and the unit empty repositioning cost in direction $a$. A similar approach can be used to verify Remark 4c.

In addition, as a numerical illustration, the relationship between the earned profit and the unit empty repositioning cost in direction $a$ (i.e., $\xi^a$) is respectively depicted in Fig. 5a to Fig. 5c for cases $\Lambda \in S_i$, $\Lambda \in S_j$ and $\Lambda \in S_j$. As these figures show, both the actual and optimal profit are respectively decreasing on average in $\xi^a$ for the case $\Lambda \in S_j$ and unchanging for the cases $\Lambda \in S_i$ and $\Lambda \in S_j$. However, the regret is unchanging for the case $\Lambda \in S_i$ and increasing for the cases $\Lambda \in S_i$ and $\Lambda \in S_j$. Note that there are some fluctuations in these figures due to the random generation of potential demands. In other words, there is some increase in both the actual and optimal profits in those cases where the average potential imbalance is larger than 40,000, i.e., $\mu^b - \mu^a$ (e.g., see $\xi^a = 200$ in Fig. 5b and $\xi^a = 550$ in Fig. 5c). Conversely, there
is a considerable decrease in those cases where the average potential imbalance is smaller than 40,000 (e.g., see $\xi_a=600$ in Fig. 5b and case $\xi_a=500$ in Fig. 5c).

![Graphs showing profit and regret variation](image)

Fig. 5. Actual profit and regret variation against the unit empty repositioning cost

**Remark 5.** In the case of uncertain potential demand, the profit of the firm is always decreasing in price sensitivity $\alpha_a$ and $\alpha_b$.

The relationship between the earned profit and the price sensitivity is intuitive in the problem under investigation. It is easy to see that any increase in the price sensitivity in one direction not only reduces the optimal price in that direction, but also diminishes the realized demand (i.e. $d'$) in that direction. Therefore, the earned profit in that direction will decrease as well. However, it does not affect the earned profit in the opposite direction as the markets are independent from each other. In addition, it is important to note that the price sensitivity may also affect the total empty repositioning cost. If the price sensitivity increases in the direction with lower realized demand (e.g. say direction $I$), it reduces the loaded movement in direction $I$ and consequently worsens the empty repositioning. These changes lead to a drop in the total profit because
of the decrease in profit of direction $I$ and the increase in the total empty repositioning cost. Conversely, if
the price sensitivity increases in the direction with more demand (e.g. say direction $II$), both the loaded
movement in direction $II$ and the total empty repositioning decrease. Although there are savings in the total
empty repositioning cost, the total profit reduction in direction $II$ outweighs this gain by lowering the total
empty repositioning costs, and thus, the profit drops. This can be mathematically proven by taking the
derivative of the profit functions with respect to price sensitivity. For instance, the derivative of the profit
function with respect to $\alpha^b$ for $\Lambda \in S$, demonstrates that the negative effect on the earned profit is more than
the positive effect on the empty repositioning cost. Consequently, any increase in price sensitivity decreases
the total earned profit.

Fig. 6 shows the effect of price sensitivity on the earned profit through a numerical example. As
depicted in this figure, the total profit is decreasing in both price sensitivity parameters. However, since the
chosen pricing policy is $P_1$ (i.e. $d^a > d^b$) the decrement rate in $\alpha^c$ (Fig. 6a) is significantly more than the one
in $\alpha^b$ (Fig. 6b). Moreover, as these figures show the regret is increasing in $\alpha^c$ when $\Lambda > 0$ whereas it is
unchanging when $\Lambda < 0$ (similar to Fig. 6b).

7. Conclusions and Future Research Avenues

Considering the impact of pricing and quality on a firm’s profit in a monopoly market, this study
developed the optimal PQS strategy for managing empty container repositioning in the presence of
uncertainty. Depending on the realized demand imbalance, three policies were defined to capture all the
loaded and empty movement costs in determining the optimal value of the price and the level of quality. To investigate the impact of demand uncertainty on these decisions, we utilized a robust optimization technique that incorporated both maximum and average regret amounts in the case of uncertain potential demand. This technique was then used to analyze three policies in terms of the regret amount that the firm experiences under different conditions. Comparing the regret amounts enabled us to define a robust PQS strategy for the firm to handle its concerns about the service quality and price. In addition, to easily implement the proposed PQS strategy, we developed the PQSASS algorithm to convert the obtained results to a relatively simple procedure that is implementable for the shipping firm. Finally, our analyses demonstrate that setting the control parameter to 0.5 results in the highest value of actual profit and the lowest value of EM-Reg. This result verifies that the PQS strategy outperforms the deterministic pricing approach that exists in the literature in terms of the firm's profitability. It also enables the shipping firms to choose an appropriate level of service quality for satisfying their customers' needs and for their overall profitability. Additionally, the developed strategy would assist shipping firms with taking demand uncertainty into consideration and being more responsive to customers' demands, which would lead to an increase in customer satisfaction in the maritime industry. Moreover, our analysis shows that the profit earned by applying the PQS strategy depends on several parameters such as the potential imbalance, demand variation coefficient, and loaded and empty repositioning costs that should be considered by the shipping firm.

Our investigations demonstrated that choosing a policy to have a balanced market does not necessarily result in a larger profit. Another important insight is that increasing the service quality level from RS to QS does not necessarily improve the firm’s profit. To determine the optimal level of quality, the firm must closely assess the impact of a higher service quality on both the realized demand and the cost of service (i.e., an additional loaded movement cost). To this end, we have developed several conditions that can assist a firm with determining its optimal quality level. We have also observed that the optimal PQS strategy in an uncertain environment results in a larger profit compared to minimizing either the maximum regret or the average regret.
Despite investigating demand uncertainty and the impact of service quality in our analysis, our study has several limitations. First, for the sake of analytical investigations, we considered only two levels of service quality. Although the presented analysis could assist firms in deciding whether or not to offer value-added services to their clients, it may not be viewed as comprehensive enough to be used by all carriers. This is because some carriers may focus on selecting a specific value-added service from a wide range of additional services based on a combination of transit time, reliability, and flexibility in pickup and delivery services. Furthermore, we developed a robust PQS strategy for a monopoly market to simplify the complicated conditions. Although this PQS strategy is practical for strategy alliances, it could not be used in competing markets with two or more shipping companies. In addition, the current study considers a single transportation service system in which all the containers are assumed to be homogeneous. However, there are variant sizes of containers with different movement costs and service prices in some maritime transportation systems. Last, but not least, this study investigated a two-port transportation system to obtain closed-form solutions, which might be a simpler system compared to complex real-world transportation systems. Consequently, the results obtained in this study are not necessarily applicable for a multi-demand transportation system requiring routing decisions in addition to pricing.

Regarding the limitations mentioned, this study can be further extended to explore other interesting research avenues. Since the focus of this study was on a single firm and single service transportation system, it does not consider rival competition in the market. Hence, developing a robust PQS strategy in a transportation system, in which two or more carriers are competing to provide service, empty containers are repositioned between demand nodes due to demand imbalance, and the demand is uncertain, could be an interesting topic. In addition, developing a robust PQS strategy for a multi-service transportation system could be an avenue for future study. As mentioned, the current study assumed all the containers were the homogenous. Due to the availability of different sizes of containers in maritime transportation systems, it would be interesting to generalize the developed PQS strategy for a multi-service transportation system with different container sizes. Lastly, as stated previously, there are several shipping firms that provide transportation service between three or more ports. These transportation systems are more complicated than
the system investigated in the current study, and therefore, they could provide an interesting research opportunity.

References


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Zhang R., Huang C., Feng X. (2020). Empty container repositioning with foldable containers in a river


APPENDICES

APPENDIX A. NOTATIONS

A-1. DETERMINISTIC CASE

\( s \): Indicates a specific direction between two nodes, \( s \in \{a, b\} \);

\( D^* \): Potential demand volume in direction \( s \) in the deterministic case;

\( \bar{d}^* \): Realized demand volume in direction \( s \) in the deterministic case;

\( \alpha^* \): The price sensitivity, measuring the demand responsiveness to the price in direction \( s \);

\( \beta^* \): The quality sensitivity, measuring the demand responsiveness to the level of service quality in direction \( s \);

\( c^* \): The loaded movement cost in direction \( s \);

\( \omega^* \): The amount of growth in the loaded movement cost of direction \( s \) if the firm chooses the higher level of service quality;

\( \xi^* \): The empty repositioning cost in direction \( s \);

\( T_a^U \): A threshold used to distinguish state \( S_1 \);

\( T_a^L \): A threshold used to distinguish state \( S_2 \);

\( \Delta D \): Potential imbalance indicating the difference between potential demand in direction \( a \) and \( b \);

\( p^* \): The unit price of transportation service in direction \( s \);

\( q \): 1 if the higher level of quality is chosen for transportation service; 0 otherwise;

\( \Pi \): A variable indicating the profit gained by the company throughout the planning horizon;

\( \Pi_0 \): A variable indicating the profit gained by the company at RS service quality;

\( \Pi_1 \): A variable indicating the profit gained by the company at QS service quality;

\( \nu^* \): Quality performance indicator;

\( \theta^* \): A dependent variable that is equal to 1 if \( \nu^* \) is positive, and 0 if \( \nu^* \) is negative.

A-2. UNCERTAIN CASE

\( s \): Indicates a specific direction between two nodes, \( s \in \{a, b\} \);

\( D^*_a \): The lower level of potential demand in direction \( s \);

\( D^*_a \): The upper level of potential demand in direction \( s \);

\( \bar{D}^* \): A uniform random variable showing the potential demand volume in direction \( s \), \( \bar{D}^* \in [D^*_a, D^*_a] \);
\( \hat{\theta}^s \): The materialized value of \textit{potential} demand in direction \( s \);

\( \bar{\hat{\theta}}^s \): A point estimation for \( \hat{\theta}^s \) used in pricing and quality setting;

\( \tilde{d}^s \): The volume of \textit{realized} demand in direction \( s \) under uncertainty;

\( \alpha^s \): The price sensitivity, measuring the demand responsiveness to the price in direction \( s \);

\( \beta^s \): The quality sensitivity, measuring the demand responsiveness to the level of service quality in direction \( s \);

\( c^s \): The loaded movement cost in direction \( s \);

\( \omega^s \): The amount of growth in the loaded movement cost of direction \( s \) if the firm chooses the higher level of service quality;

\( \xi^s \): The empty repositioning cost in direction \( s \);

\( T^u_{\theta} \): A threshold used to distinguish state \( S_i \);

\( T^u_{\eta} \): A threshold used to distinguish state \( S_2 \);

\( \lambda \): The pessimism control parameter;

\( \hat{\Lambda} \): The materialized value of the potential imbalance;

\( \Delta^s \): Difference between \( \hat{\theta}^s \) and \( \bar{\hat{\theta}}^s \) (i.e., \( \hat{\theta}^s - \bar{\hat{\theta}}^s \));

\( \mu^s \): Expected value of potential demand in direction \( s \), \( \mu^s = E(\hat{\theta}^s) \);

\( \kappa \): Variation coefficient for potential demand, \( \hat{\theta}^s \in [\mu^s - \kappa \mu^s, \mu^s + \kappa \mu^s] \);

\( R^s \): Range of variation for potential demand in direction \( s \), \( R^s = \tilde{d}^s - \hat{\theta}^s \).

\( p^s \): The unit price of transportation service in direction \( s \);

\( q \): 1 if the higher level of quality is chosen for transportation service; 0 otherwise;

\( \Pi \): A variable indicating the profit gained by the company throughout the planning horizon;

\( \Pi_0 \): A variable indicating the profit gained by the company at RS service quality;

\( \Pi_1 \): A variable indicating the profit gained by the company at QS service quality;

\( \nu^s \): Quality performance indicator;

\( \theta^s \): A dependent variable that is equal to 1 if \( \nu^s \) is positive, and 0 if \( \nu^s \) is negative;

\( \mathfrak{R} \): A variable indicating the anticipated regret;

\( \mathfrak{N} \): Called EM-Reg it is a function that combines the expected value and the maximum regret.
A-3. Parameter Settings in Numerical Analyses

As discussed in the manuscript, the notations can be categorized into two groups: parameters and variables. To conduct the numerical analyses and compare obtained results with the ones in the literature, the parameters have been selected from the literature, more specifically, from Zhou and Lee (2009), and Chen et al. (2016). Table A-1 shows the range of variations for these parameters in our numerical analyses. The second category includes several dependent and independent variables that are determined after employing the developed mechanism in the current study.

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<th>Range of Variation</th>
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<tr>
<td>Expected Value of Potential Demand</td>
<td>$\mu$</td>
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<tr>
<td>Coefficient of Demand Variation</td>
<td>$\kappa$</td>
<td>0.0 – 0.5</td>
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<td>$\Lambda$</td>
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</tr>
<tr>
<td>Empty Repositioning Cost</td>
<td>$\xi$</td>
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<tr>
<td>Price Sensitivity</td>
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<td>The QS Service Extra Cost</td>
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APPENDIX B. PROOF OF PROPOSITION 1

As mentioned, the relation between the realized demand and the potential demand in two directions is defined as: \( d^a = D^a - \alpha^a p^a + \beta^a q \) and \( d^b = D^b - \alpha^b p^b + \beta^b q \). In addition, the firm aims to maximize function \( \Pi = (p^a - c^a - \omega^a q)d^a + (p^b - c^b - \omega^b q)d^b - \xi^a (d^a - d^a)^+ - \xi^b (d^b - d^b)^+ \). To simplify the formulation, we define these two linearized profit functions:

\[
\tilde{\pi} = (p^a - c^a - \omega^a q\xi^b)d^a + (p^b - c^b - \omega^b q\xi^a)d^b, \quad \text{where} \quad d^a \geq d^b
\]

and

\[
\pi = (p^a - c^a - \omega^a q\xi^b)d^a + (p^b - c^b - \omega^b q\xi^a)d^b, \quad \text{where} \quad d^a \leq d^b.
\]

It is intuitive that \( \Pi = \max(\tilde{\pi}, \pi) \). Let \( p^*_{a} \) and \( p^*_{b} \) respectively denote the optimal price in directions \( a \) and \( b \), \( q^* \) denotes the optimal level of service quality, and \( d^*_{a} \) and \( d^*_{b} \) respectively indicate the value of demand in directions \( a \) and \( b \) realized by the set prices and quality. Clearly, the optimal value of decision variables results in one of the following cases for empty repositioning.

Case A: Empty Repositioning in Direction \( b \) (\( d^*_{a} \geq d^*_{b} \))

In this case, \( \tilde{\pi} \geq \pi \) and \( \Pi = \tilde{\pi} \). Therefore, the optimal value of prices can be obtained as follows,

\[
\tilde{\pi} = (p^a - c^a - \omega^a q - \xi^b)d^a + (p^b - c^b - \omega^b q + \xi^a)d^b = (p^a - c^a - \omega^a q - \xi^b)(D^a - \alpha^a p^a + \beta^a q) + (p^b - c^b - \omega^b q + \xi^a)(D^b - \alpha^b p^b + \beta^b q).
\]

Deriving the profit function with respect to the price parameter in direction \( a \) results in:

\[
\frac{\partial \tilde{\pi}}{\partial p^a} = (D^a - \alpha^a p^a + \beta^a q) - \alpha^a \left( p^a - c^a - \omega^a q - \xi^b \right) = 0 \Rightarrow
\]

\[
2\alpha^a p^a = D^a + \beta^a q + \alpha^a \left( c^a + \omega^a q + \xi^b \right) \Rightarrow p^*_{a} = \frac{D^a + \beta^a q + \alpha^a \left( c^a + \omega^a q + \xi^b \right)}{2\alpha^a}
\]

Hence, the value of the realized demand in direction \( a \) is obtained as follows

\[
d^*_{a} = D^a - \alpha^a \left( \frac{D^a + \beta^a q + \alpha^a \left( c^a + \omega^a q + \xi^b \right)}{2\alpha^a} \right) + \beta^a q = \frac{1}{2}(D^a + \beta^a q - \alpha^a \left( c^a + \omega^a q + \xi^b \right)).
\]

Similarly,
\[
\frac{\partial \pi}{\partial p^a} = (D^a - \alpha^a p^a + \beta^a q^a) - \alpha^a (p^a - c^a - \omega^a q^a + \xi^a) = 0 \Rightarrow \\
2\alpha^a p^a = D^a + \beta^a q^a + \alpha^a (c^a + \omega^a q^a - \xi^a) 
\]
\[
p^a = \frac{D^a + \beta^a q^a + c^a + \omega^a q^a - \xi^a}{2\alpha^a} 
\]
and the value of the realized demand in direction \( b \) is
\[
d^b = D^b - \alpha^b \left( \frac{D^b + \beta^b q^b + c^b + \omega^b q^b - \xi^b}{2\alpha^b} \right) + \beta^b q^b = \frac{1}{2} \left( D^b + \beta^b q^b - \alpha^b (c^b + \omega^b q^b - \xi^b) \right).
\] (B-7)

Therefore, \( \bar{\pi}^* \) is achieved as follows
\[
\bar{\pi}^* = \left( \frac{D^a + \beta^a q^a + c^a + \omega^a q^a + \xi^a}{2\alpha^a} \right)^2 + \left( \frac{D^b + \beta^b q^b - \alpha^b (c^b + \omega^b q^b - \xi^b)}{2\alpha^b} \right)^2.
\] (B-8)

**Case B: Empty Repositioning in Direction \( a \) (\( d^a \leq d^b \))**

Unlike case A, in this case, \( \bar{\pi} \leq \pi \) and \( \Pi^* = \pi^* \). Therefore, the optimal value of the prices can be obtained as follows,
\[
\bar{\pi} = (p^a - c^a - \omega^a q^a + \xi^a) d^a + (p^b - c^b - \omega^b q^b - \xi^b) d^b = (p^a - c^a - \omega^a q^a + \xi^a) \left( D^a - \alpha^a p^a + \beta^a q^a \right) + (p^b - c^b - \omega^b q^b - \xi^b) \left( D^b - \alpha^b p^b + \beta^b q^b \right)
\] (B-9)

Now,
\[
\frac{\partial \bar{\pi}}{\partial p^a} = (D^a - \alpha^a p^a + \beta^a q^a) - \alpha^a (p^a - c^a - \omega^a q^a + \xi^a) = 0 \Rightarrow \\
2\alpha^a p^a = D^a + \beta^a q^a + \alpha^a (c^a + \omega^a q^a - \xi^a) 
\]
\[
p^a = \frac{D^a + \beta^a q^a + c^a + \omega^a q^a - \xi^a}{2\alpha^a} 
\] (B-10)

Hence, the value of the realized demand in direction \( a \) is obtained as follows
\[
d^a = D^a - \alpha^a \left( \frac{D^a + \beta^a q^a + c^a + \omega^a q^a - \xi^a}{2\alpha^a} \right) + \beta^a q^a = \frac{1}{2} \left( D^a + \beta^a q^a - \alpha^a (c^a + \omega^a q^a - \xi^a) \right).
\] (B-11)

Similarly,
\[
\frac{\partial \bar{\pi}}{\partial p^b} = (D^b - \alpha^b p^b + \beta^b q^b) - \alpha^b (p^b - c^b - \omega^b q^b - \xi^b) = 0 \Rightarrow \\
2\alpha^b p^b = D^b + \beta^b q^b + \alpha^b (c^b + \omega^b q^b + \xi^b) 
\]
\[
p^b = \frac{D^b + \beta^b q^b + c^b + \omega^b q^b + \xi^b}{2\alpha^b} 
\] (B-12)
and the value of the realized demand in direction $b$ is

$$d^b = D^b - \alpha^b \left( \frac{D^b + \beta^b q}{2\alpha^b} + \frac{c^b + \omega^b q + \xi^a}{2} \right) + \beta^b q = \frac{1}{2} \left( D^b + \beta^b q - \alpha^b \left( c^b + \omega^b q + \xi^a \right) \right).$$ (B-13)

Therefore, $\pi^*$ is achieved as follows

$$\pi^* = \frac{\left( \frac{D^b + \beta^b q}{2\alpha^b} + \frac{c^b + \omega^b q + \xi^a}{2} \right)}{2} \left( D^a + \beta^a q - \alpha^a \left( c^a + \omega^a q + \xi^a \right) \right)$$

$$+ \frac{\left( \frac{D^b + \beta^b q}{2\alpha^b} + \frac{c^b + \omega^b q + \xi^a}{2} \right)}{2} \left( D^b + \beta^b q - \alpha^b \left( c^b + \omega^b q + \xi^a \right) \right)$$

$$= \frac{(D^b + \beta^b q - \alpha^a \left( c^a + \omega^a q + \xi^a \right))^2 + (D^b + \beta^b q - \alpha^b \left( c^b + \omega^b q + \xi^a \right))^2}{4\alpha^a}.$$ (B-14)

**Case C: No Empty Repositioning ($d^a = d^b$)**

In this case, setting the prices and service quality results in the same realized demand in both directions.

Therefore, no empty repositioning has occurred in the system. Under this condition,

$$D^a - \alpha^a p^a + \beta^a q = D^b - \alpha^b p^b + \beta^b q \Rightarrow \alpha^a p^a = D^a - \alpha^a p^a + \beta^a q - D^b - \beta^b q$$

$$\Rightarrow p^a = \frac{D^a - \alpha^a p^a + \beta^a q - D^b - \beta^b q}{\alpha^a}.$$ (B-15)

Replacing Equation (B-15) in the profit function results in,

$$\pi = \left( p^a - c^a - \omega^a q \right) d^a + \left( p^b - c^b - \omega^b q \right) d^b = \left( p^a - c^a - \omega^a q \right) d^a + \left( p^b - c^b - \omega^b q \right) d^b$$

$$= \left( p^a - c^a - \omega^a q + \frac{D^a - \alpha^a p^a + \beta^a q - D^b - \beta^b q}{\alpha^a} \right) \left( \frac{D^b + \beta^b q}{2\alpha^b} + \frac{c^b + \omega^b q + \xi^a}{2} \right) \left( D^b + \beta^b q - \alpha^b \left( c^b + \omega^b q + \xi^a \right) \right)$$

To achieve the best value of the price, similar to previous cases, the partial derivative of the profit function, in respect to price variables, should be obtained and set to zero.

$$\frac{\partial \pi}{\partial p^a} = \left( 1 + \frac{\alpha^a}{\alpha^b} \right) \left( D^a - \alpha^a p^a + \beta^a q \right) - \alpha^a \left( p^a - c^a - \omega^a q + \frac{D^a - \alpha^a p^a + \beta^a q - D^b - \beta^b q}{\alpha^b} - c^a - \omega^a q \right) = 0$$ (B-17)

Therefore,

$$p^a = \frac{D^a + \beta^a q}{2\alpha^a} + \frac{\left( D^a + \beta^a q \right) - \left( D^b + \beta^b q \right) + \alpha^b \left( c^a + \omega^a q + c^b + \omega^b q \right)}{2 \left( \alpha^a + \alpha^b \right)}.$$ (B-18)

Similarly,

$$p^b = \frac{D^b + \beta^b q}{2\alpha^b} + \frac{\left( D^b + \beta^b q \right) - \left( D^a + \beta^a q \right) + \alpha^a \left( c^a + \omega^a q + c^b + \omega^b q \right)}{2 \left( \alpha^a + \alpha^b \right)}.$$ (B-19)

In addition, the value of the realized demand is obtained as follows:
\[ d^a = d^b = \frac{\alpha^a (D^b + \beta^a q) + \alpha^b (D^a + \beta^b q) - \alpha^a \alpha^b (c^a + \omega^a q + c^b + \omega^b q)}{2(\alpha^a + \alpha^b)} \]  

and \( \Pi^* \) is achieved as follows

\[ \Pi^* = \frac{\left( \alpha^a (D^b + \beta^a q) + \alpha^b (D^a + \beta^b q) - \alpha^a \alpha^b (c^a + \omega^a q + c^b + \omega^b q) \right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} \]  

Since each case results in a different value of the price, we titled them differently (i.e. pricing policy \( P_1 \) for Case A, pricing policy \( P_2 \) for Case B, and pricing policy \( P_3 \) for Case C). In order to determine which one is the best policy regarding the problem condition, we should analyze the boundary conditions as follows:

\[ d^a \geq d^b \Rightarrow \frac{1}{2} \left( D^a + \beta^a q - \alpha^a (c^a + \omega^a q + \xi^a) \right) \geq \frac{1}{2} \left( D^b + \beta^b q - \alpha^b (c^b + \omega^b q - \xi^b) \right) \]  

\[ D^a - D^b \geq \frac{\alpha^a (c^a + \omega^a q + \xi^a) - \alpha^b (c^b + \omega^b q - \xi^b) - (\beta^a - \beta^b) q}{\xi^a} \]  

Similarly,

\[ d^a \leq d^b \Rightarrow \frac{1}{2} \left( D^a + \beta^a q - \alpha^a (c^a + \omega^a q - \xi^a) \right) \geq \frac{1}{2} \left( D^b + \beta^b q - \alpha^b (c^b + \omega^b q + \xi^b) \right) \]  

\[ D^a - D^b \leq \frac{\alpha^a (c^a + \omega^a q - \xi^a) - \alpha^b (c^b + \omega^b q + \xi^b) - (\beta^a - \beta^b) q}{\xi^a} \]  

Hence, the optimal pricing policy is respectively \( P_1, P_2, \) and \( P_3 \) if \( D^a - D^b \geq T_j^u \), \( D^a - D^b \leq T_j^l \) and

\[ T_j^l < D^a - D^b < T_j^u \].□
APPENDIX C. PROOF OF PROPOSITION 2

To determine the optimal level of the quality variable, we first rewrite the optimal value of the earned profit in each case. Then, we analyze its relation to the quality variable to optimize the service quality.

**Case-A.** In this case, the profit function can be rewritten as

\[
\bar{\pi}^* = \frac{\left( D^a - \alpha^a \left( c^a + \xi^a \right) + q \left( \beta^a - \alpha^a \omega^a \right) \right)^2}{4\alpha^a} + \frac{\left( D^b - \alpha^b \left( c^b - \xi^b \right) + q \left( \beta^b - \alpha^b \omega^b \right) \right)^2}{4\alpha^b} \quad (C-1)
\]

Let define \( \nu' = \beta^a - \alpha^a \omega^a \). Therefore,

\[
\bar{\pi}^* = \frac{\left( D^a - \alpha^a \left( c^a + \xi^a \right) + q\nu' \right)^2}{4\alpha^a} + \frac{\left( D^b - \alpha^b \left( c^b - \xi^b \right) + q\nu' \right)^2}{4\alpha^b} \quad (C-2)
\]

It is obvious that the higher level of quality results in a higher value of profit if \( \nu' \geq 0 \) and \( \nu' \geq 0 \). Moreover, the higher level of service leads to a lower value of profit if \( \nu' \leq 0 \) and \( \nu' \leq 0 \). Otherwise, Equation (C-3) can be used to determine whether the higher level of quality results in a higher value of profit or not.

\[
\Omega_q^{ax} = \bar{\pi}_{\nu=1}^* - \bar{\pi}_{\nu=0}^* = \frac{2\left( D^a - \alpha^a \left( c^a + \xi^a \right) \right)\nu^2 + \left( \nu' \right)^2}{4\alpha^a} + \frac{2\left( D^b - \alpha^b \left( c^b - \xi^b \right) \right)\nu^2 + \left( \nu' \right)^2}{4\alpha^b} \geq 0 \quad (C-3)
\]

It is intuitive that the higher level of quality brings about more profit if \( \Omega_q^{ax} \) is positive. Regarding the obtained results, the optimal value of the quality decision in case A is defined by Equation (C-4).

\[
q^* = \begin{cases} 
1 & \text{if } \nu' \geq 0 \text{ and } \nu' \geq 0 \text{ or } \nu' < 0 \text{ and } \Omega_q^{ax} > 0 \\
0 & \text{if } \nu' < 0 \text{ and } \nu' < 0 \text{ or } \nu' > 0 \text{ and } \Omega_q^{ax} < 0 
\end{cases} \quad (C-4)
\]

**Case B:** In this case, the profit function can be rewritten as

\[
\bar{\pi}^* = \frac{\left( D^a - \alpha^a \left( c^a - \xi^a \right) + q\nu' \right)^2}{4\alpha^a} + \frac{\left( D^b - \alpha^b \left( c^b + \xi^b \right) + q\nu' \right)^2}{4\alpha^b} \quad (C-5)
\]

\[
\Omega_q^{bx} = \bar{\pi}_{\nu=1}^* - \bar{\pi}_{\nu=0}^* = \frac{2\left( D^a - \alpha^a \left( c^a - \xi^a \right) \right)\nu^2 + \left( \nu' \right)^2}{4\alpha^a} + \frac{2\left( D^b - \alpha^b \left( c^b + \xi^b \right) \right)\nu^2 + \left( \nu' \right)^2}{4\alpha^b} \geq 0 \quad (C-6)
\]

Similar to case A, in this case, the optimal value of the quality decision is defined by Equation (C-7):

\[
q^* = \begin{cases} 
1 & \text{if } \nu' \geq 0 \text{ and } \nu' \geq 0 \text{ or } \nu' < 0 \text{ and } \Omega_q^{bx} > 0 \\
0 & \text{if } \nu' < 0 \text{ and } \nu' < 0 \text{ or } \nu' > 0 \text{ and } \Omega_q^{bx} < 0 
\end{cases} \quad (C-7)
\]

**Case C.** In this case, the profit function can be rewritten as
\[
\pi^* = \frac{\left( \alpha^a \left( D^a - \alpha^b c^a + \nu^a q \right) + \alpha^b \left( D^b - \alpha^a c^b + \nu^b q \right) \right)^2}{4\alpha^a \alpha^b \left( \alpha^a + \alpha^b \right)}
\] (C-8)

Therefore,
\[
\Omega_q^{\pi} = \pi^* - \pi^0 = \frac{2 \left( D^a - \alpha^a c^a \right) u^a + \left( u^a \right)^2 + 2 \left( D^b - \alpha^b c^b \right) u^b + \left( u^b \right)^2}{4\alpha^a}
\] (C-9)

Since determining the optimum value of \( q \) depends on the chosen pricing policy, we define an upper limit and a lower limit for \( \Omega_q^{\pi} \) to simplify the decision making process. Note that these upper and lower limits can be obtained easily by comparing Equations (C-3) and (C-6).
\[
\Omega_q^{\pi} \leq \min \left\{ \Omega_q^{\pi^a}, \Omega_q^{\pi^b}, \Omega_q^{\pi\pi} \right\}
\]
\[
\theta' \text{ equals } 1 \text{ if } \nu' \text{ is positive, and } 0 \text{ if } \nu' \text{ is negative. Let }
\]
\[
k^a_q = D^a - \alpha^a \left( c^a - (1 - \theta') \xi^a + \theta' \xi^b \right) \text{ and } k^b_q = D^b - \alpha^b \left( c^b + \theta' \xi^b - (1 - \theta') \xi^b \right). \text{ Therefore,}
\]
\[
\Omega_q^{\pi^a} = \frac{2k^a_q u^a + \left( u^a \right)^2}{4\alpha^a} + \frac{2k^b_q u^b + \left( u^b \right)^2}{4\alpha^b}
\] (C-11)

Similarly, the upper bound can be obtained as follows:
\[
\Omega_q^{\pi^b} = \frac{2k^a_q u^a + \left( u^a \right)^2}{4\alpha^a} + \frac{2k^b_q u^b + \left( u^b \right)^2}{4\alpha^b}
\] (C-12)

where \( k^a_q = D^a - \alpha^a \left( c^a - \theta' \xi^a + (1 - \theta') \xi^b \right) \) and \( k^b_q = D^b - \alpha^b \left( c^b + (1 - \theta') \xi^b - \theta' \xi^b \right). \) □
APPENDIX D. PROOF OF PROPOSITION 3 & COROLLARY 1

PROOF OF PROPOSITION 3:
It is intuitive that the total value of EM-Reg is the sum of the partial values in directions $a$ and $b$ (i.e. $N = N^a + N^b$). According to the definition of the EM-Reg function,

$$N^a = \lambda E \left( \mathbf{r}^a \right) + (1-\lambda) \max \left( \mathbf{r}^a \right)$$  \hspace{1cm} (D-1)$$

and

$$N^b = \lambda E \left( \mathbf{r}^b \right) + (1-\lambda) \max \left( \mathbf{r}^b \right).$$  \hspace{1cm} (D-2)$$

In addition, regarding the definition of regret, we will have

$$\mathbf{r} = \mathbf{\pi}_p - \mathbf{H}_p = (\pi_p^a + \pi_p^b) - \left[ (p^a - c^a - \omega^a q) d^a + (p^b - c^b - \omega^b q) d^b - \xi^a (d^a - d^*)^+ - \xi^b (d^b - d^*)^+ \right]$$

$$\mathbf{r} = r^a + r^b$$  \hspace{1cm} (D-3)$$

In order to minimize $\mathbf{r}^a$, we should respectively find the best value of $p^a$ and $p^b$ that conclude the minimum values for $\mathbf{r}^a$ and $\mathbf{r}^b$. To minimize the value of $\mathbf{r}^a$,

$$\mathbf{r}^a = \pi_p^a - \left( p^a - c^a - \omega^a q - \xi^b \right) d^a =$$

$$= \frac{D^a + \beta^b q - \alpha^a (c^a + \omega^a q + \xi^b)}{4\alpha^a} - \left( p^a - c^a - \omega^a q - \xi^b \right) \left( \frac{D^a - \alpha^a p^a + \beta^a q}{4\alpha^a} \right)$$

$$= \frac{(D^a + E)^2}{4\alpha^a} - A(D^a + B)$$  \hspace{1cm} (D-4)$$

Since the value of the potential demand is uncertain, the expected value of regret in direction $a$ equals to:

$$E \left( \mathbf{r}^a \right) = \int_{D_n}^{D_n} \mathbf{r}^a \mid f_D\left( x \right) dx = \int_{D_n}^{D_n} \left( \frac{x + E}{4\alpha^a} - A(x + B) \right) \left( \frac{1}{D_n - D_n} \right) dx$$

$$= \left( \frac{1}{D_n - D_n} \right) \left( \frac{1}{12\alpha^a} (x + E)^3 - \frac{1}{2} A x^2 - A B x \right) \left( \frac{1}{D_n - D_n} \right)$$

$$= \frac{(D_n + E)^2 + (D_n + E)^2 + (D_n + E)(D_n + E)}{12\alpha^a} - \left( \frac{1}{2} A(D_n + D_n) + A B \right).$$  \hspace{1cm} (D-5)$$

In addition, the maximum value of regret in direction $a$ can be calculated as follows:

$$\max \left( \mathbf{r}^a \right) = \max \left\{ \left( \frac{x + E}{4\alpha^a} - A(x + B) \mid x \in [D_n, D_n] \right) \right\}$$  \hspace{1cm} (D-6)$$

It is intuitive that replacing the first demand variable with the maximum potential demand and the second demand variable with the minimum value of potential demand results in a maximum value for $\mathbf{r}^a$. Hence,
\[ \operatorname{Max}(\mathbf{R}^e) = \frac{(D_n^e + E)^2}{4\alpha^e} - A(D_n^e + B) \] \hspace{1cm} \text{(D-7)}

Consequently, the EM-Reg in direction \( a \) can be calculated as:

\[ N^a = \lambda E(\mathbf{R}^e) + (1 - \lambda) \operatorname{Max}(\mathbf{R}^e) = \lambda \left( \frac{(D_n^e + E)^2 + (D_n^e + E)^2}{12\alpha^e} + \frac{1}{2} A(D_n^e + D_n^e) - AB \right) + (1 - \lambda) \left( \frac{(D_n^e + E)^2}{4\alpha^e} - A(D_n^e + B) \right) \] \hspace{1cm} \text{(D-8)}

Since \( \frac{\partial A}{\partial p^a} = 1 \), \( \frac{\partial B}{\partial p^a} = -\alpha^a \) and \( \frac{\partial E}{\partial p^a} = 0 \), we will have

\[ \frac{\partial N^a}{\partial p^a} = 0 \Rightarrow \lambda \left( -\frac{D_n^e + D_n^e}{2} \right) - (1 - \lambda) (D_n^e + B) - \alpha^a \lambda A = 0 \]
\[ \frac{\lambda(D_n^e + D_n^e)}{2} + \lambda B - \alpha^a \lambda A + (1 - \lambda) D_n^e + (1 - \lambda) B - \alpha^a (1 - \lambda) A = 0 \Rightarrow \]
\[ D_n^e + \frac{\lambda}{2} (D_n^e - D_n^e) + B - \alpha^a A = 0 \Rightarrow D_n^e + \frac{\lambda}{2} (D_n^e - D_n^e) - \alpha^a p^a + \beta^a q^a - \alpha^a (p^a - c^a - \omega^a q^a - \xi^a) = 0 \]
\[ p^a = \left[ D_n^e + \frac{\lambda}{2} (D_n^e - D_n^e) \right] + \frac{\beta^a q^a}{2\alpha^a} + \frac{c^a + \omega^a q^a + \xi^a}{2} \] \hspace{1cm} \text{(D-9)}

Hence, to minimize the EM-Reg function, the potential demand of direction \( a \) should be set to equal the value obtained by Equation (D-10) in pricing policy \( P_1 \).

\[ \bar{D}_n^e = D_n^e + \frac{\lambda}{2} (D_n^e - D_n^e) \] \hspace{1cm} \text{(D-10)}

Similar results can be achieved for \( \bar{D}_n^b \). Hence,

\[ \bar{D}_n^b = D_n^b + \frac{\lambda}{2} (D_n^b - D_n^b) \] \hspace{1cm} \text{(D-11)}

Finally, this approach can be extended for pricing policies \( P_2 \) and \( P_3 \) to obtain the same results. \( \square \)

**Proof of Corollary 1:**

To minimize the expected value of regret we need to set \( \lambda = 1 \). In this case, the value of the potential demands (i.e. \( D^a \) and \( D^b \)) is respectively equal:

\[ \bar{D}_n^e = D_n^e + \frac{1}{2} (D_n^e - D_n^e) = \frac{1}{2} (D_n^e + D_n^e) = E(\bar{D}_n^e), \] \hspace{1cm} \text{(D-12)}
\[ \bar{D}_n^b = D_n^b + \frac{1}{2} (D_n^b - D_n^b) = \frac{1}{2} (D_n^b + D_n^b) = E(\bar{D}_n^b). \] \hspace{1cm} \text{(D-13)} \( \square \)
APPENDIX E. PROOF OF PROPOSITION 4

Assume that \( Z_1 \) and \( Z_2 \) follow a uniform distribution as follows:

\[
Z_1 \sim \text{Unif} \left[ D_{a}^r, D_{a}^b \right] \quad Z_2 \sim \text{Unif} \left[ D_{b}^r, D_{b}^b \right]
\]  
(E-1)

Then \( P(Z_1 - Z_2 \geq T) \) can be calculated as

\[
p_1 = P(Z_1 - Z_2 \geq T) = P(Z_2 \leq Z_1 - T) = \int_{z_2=-\infty}^{\min\{z_2,T\}} \int_{z_1=-\infty}^{z_1+T} \frac{1}{R^a R^b} dz_1 dz_2
\]  
(E-2)

Since both variables follow a uniform distribution, the value of Equation (E-2) is equal to the hatched area in Fig. E1 regarding the values of \( D_{a}^r, D_{a}^b, D_{b}^r, \) and \( D_{b}^b \).

In addition, \( P(Z_1 - Z_2 \leq L) \) can be calculated as

\[
P(Z_1 - Z_2 \leq L) = P(Z_1 \leq Z_2 + L) = \int_{z_2=-\infty}^{\min\{z_2,L\}} \int_{z_1=-\infty}^{z_1+L} \frac{1}{R^a R^b} dz_1 dz_2
\]  
(E-3)

Fig. E1. Probability of case A (i.e. \( p_1 \)) under uncertain potential demand.
Similarly, the value of Equation (E-3) is equal to the hatched area in Fig. E2 regarding the values of $D^e_a$, $D^e_b$, $D^n_a$, and $D^n_b$.

\[
P(\Omega \in S_2) = 0
\]

\[
P(\Omega \in S_3) = \frac{(D^b_n - D^a_n + L)^2}{2R^a R^b}
\]

\[
P(\Omega \in S_4) = \frac{(D^b_n + D^a_n) - 2(D^a_n - L) \cdot (D^e_n - D^a_n - L)}{2R^a}
\]

\[
P(\Omega \in S_5) = 1 - \frac{(D^b_n - D^a_n - L)^2}{2R^a R^b}
\]

\[
P(\Omega \in S_6) = 1
\]

**Fig. E2.** Probability of case B (i.e. $p_2$) under uncertain potential demand

\[\square\]
APPENDIX F. PROOF OF PROPOSITION 5

To prove the proposition, we separately calculate the value of the regret that is imposed by adopting policy $P_1$ given that each state $S_1$, $S_2$, and $S_3$ will have occurred.

A. Regret imposed by adopting policy $P_1$ given that state $S_1$ will have occurred (i.e. $\text{Reg}(P_1 | S_1)$)

Regarding the definition of regret and the occurred state, $\text{Reg}(P_1 | S_1) = \Pi_1 - \hat{\Pi}_1$, where $\Pi_1^*$ indicates the maximum value of profit that is achievable in state $S_1$ if we know the materialized values of potential demand at the beginning of the pricing process. In addition, $\hat{\Pi}_1$ demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy $P_1$. According to Equation (4), adopting policy $P_1$ results in $p^a = \frac{\hat{D}^a + \beta^a q}{2\alpha^a} + \frac{c^a + \omega^a q + \xi^b}{2}$ and $p^b = \frac{\hat{D}^b + \beta^b q}{2\alpha^b} + \frac{c^b + \omega^b q - \xi^b}{2}$. Furthermore, the maximum achievable profit is

$$\Pi_1^* = \left(\frac{\hat{D}^a + \beta^a q - \alpha^a (c^a + \omega^a q + \xi^b)}{4\alpha^a}\right)^2 + \left(\frac{\hat{D}^b + \beta^b q - \alpha^b (c^b + \omega^b q - \xi^b)}{4\alpha^b}\right)^2.$$  \hspace{1cm} (F-1)

To calculate the total regret, it is divided into directions $a$ and $b$.

$$\text{Reg}(P_1 | S_1) = \Pi_1^* - \hat{\Pi}_1 = \left(\Pi_1^{a*} + \Pi_1^{b*}\right) - \left(\hat{\Pi}_1^{a*} + \hat{\Pi}_1^{b*}\right) = \left(\Pi_1^{a*} - \hat{\Pi}_1^{a*}\right) + \left(\Pi_1^{b*} - \hat{\Pi}_1^{b*}\right) = R_1^a + R_1^b$$  \hspace{1cm} (F-2)

$$R_1^a = \left(\frac{\hat{D}^a + \beta^a q - \alpha^a (c^a + \omega^a q + \xi^b)}{4\alpha^a}\right)^2 - \left(p^a - c^a - \omega^a q - \xi^b\right)^2 \hat{d}^a = \frac{\left(\hat{T}_1^a\right)^2}{4\alpha^a} - \left(p^a - c^a - \omega^a q - \xi^b\right)^2 \hat{d}^a$$  \hspace{1cm} (F-3)

where $\hat{T}_1 = \hat{D}^a + \beta^a q - \alpha^a (c^a + \omega^a q + \xi^b)$. In addition,

$$\hat{\Pi}_1^{a*} = \left(p^a - c^a - \omega^a q - \xi^b\right) \hat{d}^a = \left(\frac{\hat{D}^a + \beta^a q}{2\alpha^a} + \frac{c^a + \omega^a q + \xi^b}{2} - (c^a + \omega^a q + \xi^b)\right) \hat{d}^a$$  \hspace{1cm} (F-4)

and,

$$\hat{d}^a = \hat{D}^a - \alpha^a p^a + \beta^a q = \hat{D}^a - \alpha^a \left(\frac{\hat{D}^a + \beta^a q}{2\alpha^a} + \frac{c^a + \omega^a q + \xi^b}{2}\right) + \beta^a q$$

$$= \frac{2\hat{D}^a - \hat{D}^a + \beta^a q}{2} - \frac{\alpha^a}{2} \left(c^a + \omega^a q + \xi^b\right) = \frac{\Delta^a + \hat{D}^a + \beta^a q - \alpha^a (c^a + \omega^a q + \xi^b)}{2}$$  \hspace{1cm} (F-5)

Hence,
\[
\hat{H}_i = (p^* - c^* - \omega_q - \xi^b) \hat{a} = \left( \frac{\hat{D}^e + \hat{D}^s - 2\alpha^e - (c^* + \omega_q + \xi^b)}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e - (c^* + \omega_q + \xi^b)}{2} \right) \\
= \left( \frac{\hat{D}^e + \hat{D}^s - 2\alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^e + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \\
= \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right) \left( \frac{\hat{D}^* + \hat{D}^s - \alpha^e}{2} \right)
\]

where \( \hat{D}^e \) and \( \hat{D}^s \) are the actual demands in states \( e \) and \( s \), respectively.

Similarly, it can prove that \( R_{i b}^b = \frac{(\Delta^b)^2}{4\alpha^b} \). Hence,

\[
\text{Reg}(P_i | S_j) = R_{i a}^a + R_{i b}^b = \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}.
\]

**B. Regret imposed by adopting policy \( P_1 \) given that state \( S_2 \) will have occurred (i.e. \( \text{Reg}(P_1 | S_2) \))**

In this case, \( \text{Reg}(P_1 | S_2) = H'_2 - H'_1 \) where \( H'_1 \) indicates the maximum value of profit that is achievable in the state \( S_2 \) if we know the materialized values of potential demand at the beginning of the pricing process.

In addition, \( H'_2 \) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \( P_1 \). According to Equation (4), adopting policy \( P_1 \) results in

\[
p^* = \frac{\hat{D}^e + \hat{D}^s}{2\alpha^e} + \frac{c^* + \omega_q}{2} \quad \text{and} \quad p^b = \frac{\hat{D}^e + \hat{D}^s}{2\alpha^b} + \frac{c^* + \omega_q}{2}.
\]

Furthermore, the maximum achievable profit is

\[
H'_2 = \frac{\hat{D}^e + \hat{D}^s}{4\alpha^e} + \frac{c^* + \omega_q}{4} = \frac{\hat{D}^* + \hat{D}^s}{4\alpha^b} + \frac{c^* + \omega_q}{4}.
\]

where \( \hat{T}_1 = \hat{D}^e + \hat{D}^s - \alpha^e (c^* + \omega_q - \xi^a) \), and \( \hat{T}_2 = \hat{D}^e + \hat{D}^s - \alpha^b (c^* + \omega_q + \xi^a) \). To calculate the total regret, it is divided into directions \( a \) and \( b \).

\[
\text{Reg}(P_1 | S_2) = H'_2 - H'_1 = R_{2 a}^a + R_{2 b}^b
\]
\[
R_{1a}^{2a} = \left( \tilde{T}_{i}^{a} \right)^{2} - \frac{\bar{H}_{i}^{a}}{4\alpha^{a}} \left( p^{a} - c^{a} + \omega^{a} q + \xi^{a} \right) \tilde{a}^{a} = \left( \tilde{T}_{i}^{a} - \hat{H}_{i}^{2a} \right) (F-11)
\]

\[
\hat{H}_{i}^{2a} = \left( p^{a} - c^{a} - \omega^{a} q + \xi^{a} \right) \tilde{a}^{a} = \left( \frac{\bar{D}^{a} + \beta^{a} q + \frac{c^{a} + \omega^{a} q + \bar{z}^{b}}{2}}{2\alpha^{a}} - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} \right) \tilde{a}^{a} (F-12)
\]

Also,

\[
\tilde{a} = \tilde{D}^{a} - \alpha^{a} \left( \frac{\bar{D}^{a} + \beta^{a} q + \frac{c^{a} + \omega^{a} q + \bar{z}^{b}}{2}}{2\alpha^{a}} - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} \right) + \beta^{a} q = \frac{2\bar{D}^{a} - \bar{D}^{a} + \beta^{a} q - \frac{(c^{a} + \omega^{a} q + \bar{z}^{b})}{2}}{2} \alpha^{a} \left( c^{a} + \omega^{a} q - \xi^{a} \right)
\]

\[
\Delta^{a} = \bar{D}^{a} + \beta^{a} q - \alpha^{a} \left( c^{a} + \omega^{a} q - \xi^{a} \right) \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} \right) - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} = \Delta^{a} + \bar{D}^{a} + \beta^{a} q - \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} + \bar{z}^{b} \right)
\]

\[
\Delta^{a} - \alpha^{a} (\bar{z}^{b} + \bar{z}^{b}) + \hat{T}_{i}^{a} (F-13)
\]

\[
\tilde{a} = \tilde{D}^{a} - \alpha^{a} \left( \frac{\bar{D}^{a} + \beta^{a} q + \frac{c^{a} + \omega^{a} q + \bar{z}^{b}}{2}}{2\alpha^{a}} - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} \right)
\]

\[
\Delta^{a} - \alpha^{a} \left( \bar{z}^{b} + \bar{z}^{b} \right) + \hat{T}_{i}^{a} (F-14)
\]

Consequently,

\[
R_{i}^{2a} = \left( \tilde{T}_{i}^{a} \right)^{2} - \frac{\bar{H}_{i}^{a}}{4\alpha^{a}} \left( p^{a} - c^{a} - \omega^{a} q - \xi^{a} \right) \tilde{a}^{a} = \left( \tilde{T}_{i}^{a} - \hat{H}_{i}^{2a} \right) (F-15)
\]

Similarly,

\[
R_{i}^{2b} = \left( \tilde{T}_{i}^{a} \right)^{2} - \frac{\bar{H}_{i}^{a}}{4\alpha^{a}} \left( p^{a} - c^{a} - \omega^{a} q - \xi^{a} \right) \tilde{a}^{a} = \left( \tilde{T}_{i}^{a} - \hat{H}_{i}^{2b} \right) (F-16)
\]

and

\[
\tilde{a} = \tilde{D}^{a} - \alpha^{a} \left( \frac{\bar{D}^{a} + \beta^{a} q + \frac{c^{a} + \omega^{a} q - \bar{z}^{b}}{2}}{2\alpha^{a}} - \frac{(c^{a} + \omega^{a} q + \bar{z}^{b})}{2} \right)
\]

\[
\Delta^{a} - \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} \right) \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} \right) - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} = \Delta^{a} + \bar{D}^{a} + \beta^{a} q - \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} + \bar{z}^{b} \right)
\]

We know that,

\[
\tilde{a} = \tilde{D}^{a} - \alpha^{a} \left( \frac{\bar{D}^{a} + \beta^{a} q + \frac{c^{a} + \omega^{a} q - \bar{z}^{b}}{2}}{2\alpha^{a}} - \frac{(c^{a} + \omega^{a} q + \bar{z}^{b})}{2} \right) + \beta^{a} q = \frac{2\bar{D}^{a} - \bar{D}^{a} + \beta^{a} q - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2}}{2} \alpha^{a} \left( c^{a} + \omega^{a} q - \xi^{a} \right)
\]

\[
\Delta^{a} = \bar{D}^{a} + \beta^{a} q - \alpha^{a} \left( c^{a} + \omega^{a} q - \xi^{a} \right) \alpha^{a} \left( c^{a} + \omega^{a} q - \xi^{a} \right) - \frac{(c^{a} + \omega^{a} q - \xi^{a})}{2} = \Delta^{a} + \bar{D}^{a} + \beta^{a} q - \alpha^{a} \left( c^{a} + \omega^{a} q + \bar{z}^{b} + \bar{z}^{b} \right)
\]
\[
\frac{\Delta^a + \alpha^a (\xi^a + \xi^b) + \hat{T}_2^b}{2}
\]

Therefore,
\[
\hat{R}_{11}^{2b} = \left( \frac{\hat{D}^b + \beta^b q + c^b + \omega^b q - \xi^b}{2} \right) \left( \frac{\Delta^b + \alpha^b (\xi^a + \xi^b) + \hat{T}_2^b}{2} \right)
\]
\[
= \left( \frac{\hat{D}^b - \hat{D}_b + \beta^b q + \alpha^b (c^b + \omega^b q + \xi^b) - 2\alpha^b (c^b + \omega^b q + \xi^b)}{2\alpha^b} \right) \left( \frac{\Delta^b + \alpha^b (\xi^a + \xi^b) + \hat{T}_2^b}{2} \right)
\]
\[
= \left( \frac{\Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2^b}{2\alpha^b} \right) \left( \frac{\Delta^b + \alpha^b (\xi^a + \xi^b) + \hat{T}_2^b}{2} \right)
\]
\[
= \frac{\left( \hat{T}_2^b \right)^2 - (\Delta^b + \alpha^b (\xi^a + \xi^b))^2}{4\alpha^b}
\]

and,
\[
R_{11}^{2b} = \frac{\left( \hat{T}_2^b \right)^2 - (\Delta^b + \alpha^b (\xi^a + \xi^b))^2}{4\alpha^b} = \frac{\left( \alpha^b (\xi^a + \xi^b) \right)^2}{4\alpha^b}.
\]

Hence,
\[
\text{Reg}(P | S_3) = R_{11}^{2a} + R_{11}^{2b} = \frac{(\Delta^a - \alpha^a (\xi^a + \xi^b))^2}{4\alpha^a} + \frac{(\Delta^b + \alpha^b (\xi^a + \xi^b))^2}{4\alpha^b}.
\]

C. Regret imposed by adopting policy \( P_1 \) given that state \( S_3 \) will have occurred (i.e. \( \text{Reg}(P_1 | S_3) \))

A similar approach can be used in this case. \( \text{Reg}(P_1 | S_3) = \Pi'_1 - \Pi'_1 \), where \( \Pi'_1 \) indicates the maximum value of profit that is achievable in the state \( S_3 \) if we know the materialized values of the potential demand at the beginning of the pricing process. Also, \( \Pi'_1 \) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \( P_1 \). According to Equation (4), adopting policy \( P_1 \) results in
\[
p^o = \frac{\hat{D}^o + \beta^o q}{2\alpha^o} + \frac{c^o + \omega^o q + \xi^b}{2} = \frac{x^o + y^o + \xi^b}{2}
\]
and
Furthermore, the maximum achievable profit is

\[ \Pi = \frac{\left(\alpha^a x^a + \alpha^b x^b - \alpha^a \alpha^b (y^a + y^b)\right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} = \frac{\left(\alpha^a x^a + \alpha^b x^b - \alpha^a \alpha^b (y^a + y^b + \xi^a - \xi^b)\right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} \]

(F-24)

Where \( \hat{x}^a = \hat{D}^a + \beta^a q \), \( \hat{x}^b = \hat{D}^b + \beta^b q \), \( \hat{T}_1 = \hat{x}^a - \alpha^a (y^a + \xi^a) \), and \( \hat{T}_2 = \hat{x}^b - \alpha^b (y^b - \xi^b) \).

To calculate the total actual profit (\( \tilde{\Pi}_1 \)), we divide it into directions \( a \) and \( b \).

\[
\tilde{\Pi}_1 = \tilde{\Pi}_1^a + \tilde{\Pi}_1^b = \left(p^a - y^a\right) \hat{d}^a + \left(p^b - y^b\right) \hat{d}^b
\]

(F-25)

In this case,

\[
\hat{d}^a = \hat{D}^a - \alpha^a \left(\frac{x^a + y^a + \xi^a}{2}\right) + \beta^a q = \hat{x}^a - \alpha^a \left(\frac{y^a + \xi^a}{2}\right) = \frac{2\hat{x}^a - x^a - \alpha^a \left(\frac{y^a + \xi^a}{2}\right)}{2}
\]

(F-26)

\[
\hat{d}^b = \hat{D}^b - \alpha^b \left(\frac{x^b + y^b - \xi^b}{2}\right) + \beta^b q = \hat{x}^b - \alpha^b \left(\frac{y^b - \xi^b}{2}\right) = \frac{2\hat{x}^b - x^b - \alpha^b \left(\frac{y^b - \xi^b}{2}\right)}{2}
\]

(F-27)

where \( T_1 = x^a - \alpha^a \left(\frac{y^a + \xi^a}{2}\right) = x^a - \hat{x}^a + \hat{x}^a - \alpha^a \left(\frac{y^a + \xi^a}{2}\right) = -\Delta^a + \hat{T}_1 \). Therefore,

\[
\tilde{\Pi}_1^a = \left(\frac{T_1}{2\alpha^a} + \xi^a\right) \left(\frac{\Delta^a + \hat{T}_1}{2}\right) = \left(-\Delta^a + \hat{T}_1\right) \left(\frac{\Delta^a + \hat{T}_1}{2}\right) = \frac{\hat{T}_1 - \left(\Delta^a\right)^2}{2} + \xi^a \left(\frac{\Delta^a + \hat{T}_1}{2}\right).
\]

(F-28)

Similarly,

\[
\hat{d}^b = \hat{D}^b - \alpha^b \left(\frac{x^b + y^b - \xi^b}{2}\right) + \beta^b q = \hat{x}^b - \alpha^b \left(\frac{y^b - \xi^b}{2}\right) = \frac{2\hat{x}^b - x^b - \alpha^b \left(\frac{y^b - \xi^b}{2}\right)}{2}
\]

(F-29)

Hence,

\[
\tilde{\Pi}_1^b = \left(\frac{\Delta^b + \hat{T}_2}{2}\right) = \left(\frac{\Delta^b + \hat{T}_2}{2}\right) = \left(\frac{\Delta^b + \hat{T}_2}{2}\right).
\]

(F-28)
\[
\begin{aligned}
&= \left(\frac{x^b - \alpha^b (y^b - \xi^b)}{2\alpha^b} - \xi^b\right) \left(\frac{\Delta^a + \hat{T}_2}{2}\right) = \left(\frac{T_2}{2\alpha^a} - \xi^b\right) \left(\frac{\Delta^b + \hat{T}_2}{2}\right)
\end{aligned}
\]  
\[(F-30)\]

where \(T_2 = x^b - \alpha^b (y^b - \xi^b) = x^b - \hat{x}^b + \hat{x}^b - \alpha^b (y^b - \xi^b) = -\Delta^b + \hat{T}_2\). Therefore,

\[
\begin{aligned}
\tilde{H}_{1}^{gb} &= \left(\frac{T_2}{2\alpha^a} - \xi^b\right) \left(\frac{\Delta^a + \hat{T}_2}{2}\right) = \left(-\Delta^b + \hat{T}_2, -\xi^b\right) \left(\frac{\Delta^a + \hat{T}_2}{2}\right) = \frac{\hat{T}_2^2 - (\Delta^b)^2}{4\alpha^a} - \xi^b \left(\frac{\Delta^a + \hat{T}_2}{2}\right).
\end{aligned}
\]  
\[(F-31)\]

Now, according to Equations (F-24), (F-28) and (F-31), the corresponding regret can be calculated as follows

\[
\begin{aligned}
\tilde{H}_i^b - \hat{H}_i^b &= \left(\frac{\alpha^a \hat{T}_1 + \alpha^b \hat{T}_2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)}\right)^2 - \hat{T}_i^2 - (\Delta^a)^2 - \hat{T}_i^2 - (\Delta^b)^2 - \xi^b \left(\frac{\Delta^a + \hat{T}_1}{2} - \Delta^a + \hat{T}_2\right) \\
&= \left(\frac{\alpha^a \hat{T}_1^2 + \alpha^b \hat{T}_2^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)}\right)^2 - \hat{T}_i^2 - (\Delta^a)^2 - \hat{T}_i^2 - (\Delta^b)^2 + \left(\frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}\right) + \xi^b \left(\frac{\hat{T}_2 - \hat{T}_1}{2} + \frac{\Delta^b - \Delta^a}{2}\right) \\
&= \frac{\hat{T}_i^2 - \hat{T}_i^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \left(\frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}\right) + \xi^b \left(\frac{\hat{T}_2 - \hat{T}_1}{2} + \frac{\Delta^b - \Delta^a}{2}\right) \\
&= \frac{\hat{T}_i^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \left(\frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}\right) + \xi^b \left(\frac{\hat{T}_2 - \hat{T}_1}{2} + \frac{\Delta^b - \Delta^a}{2}\right)
\end{aligned}
\]  
\[(F-32)\]

Now, regarding **Lemma 1**, Equation (F-32) can be rewritten as

\[
\begin{aligned}
\tilde{H}_i^b - \hat{H}_i^b &= \frac{(\alpha^a + \alpha^b)^2 (\xi^b)^2}{4(\alpha^a + \alpha^b)} + \left(\frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}\right) + \xi^b \left(\frac{(\alpha^a + \alpha^b)(\xi^b)}{2} + \frac{\Delta^b - \Delta^a}{2}\right) \\
&= \frac{(\Delta^a)^2 + (\alpha^b)^2}{4\alpha^a} + \frac{(\Delta^b)^2 + (\alpha^a)^2}{4\alpha^b} + \xi^b \left(\frac{\Delta^b - \Delta^a}{2}\right)
\end{aligned}
\]  
\[(F-33)\]

**LEMMA 1**: \(\hat{T}_1 - \hat{T}_2 = -\left(\alpha^a + \alpha^b\right)\xi^b\).  

**PROOF.**

\[
\hat{T}_1 - \hat{T}_2 = \left(\hat{x}^a - \alpha^a (y^a + \xi^a)\right) - \left(\hat{x}^b - \alpha^b (y^b - \xi^b)\right) = \left(\hat{D}^a + \beta^a q - \alpha^a y^a\right) - \left(\hat{D}^b + \beta^b q - \alpha^b y^b\right) = \alpha^a + \alpha^b\xi^b
\]  
\[(F-34)\]

Since \(d^a = d^b\),

\[
d^a = d^b \Rightarrow \hat{D}^a - \alpha^a p^a + \beta^a q = \hat{D}^b - \alpha^b p^b + \beta^b q \Rightarrow \hat{D}^a + \beta^a q = \alpha^a p^a - \hat{D}^b + \beta^b q \Rightarrow \alpha^a p^a = \alpha^b p^b
\]  
\[(F-35)\]

Therefore,
\[ K_1 - K_2 = \alpha^a p^{*a} - \alpha^b p^{*b} - \alpha^a y^a + \alpha^b y^b = \alpha^a (p^{*a} - y^a) - \alpha^b (p^{*b} - y^b) \]
\[ = \alpha^a \left( \frac{\hat{x}^a - \hat{y}^a + \alpha^b (y^b + y^a) + \xi^b}{2(\alpha^a + \alpha^b)} - y^a \right) - \alpha^b \left( \frac{\hat{x}^b - \hat{y}^b + \alpha^a (y^a + y^b) + \xi^a}{2(\alpha^a + \alpha^b)} - y^b \right) \]
\[ = \frac{(2 \alpha^a + \alpha^b)(\hat{x}^a - \alpha^a y^a) - \alpha^a (\hat{x}^b - \alpha^b y^b)}{2(\alpha^a + \alpha^b)} - \frac{(2 \alpha^b + \alpha^a)(\hat{x}^b - \alpha^b y^b) - \alpha^b (\hat{x}^a - \alpha^a y^a)}{2(\alpha^a + \alpha^b)} \]
\[ = \left[ \frac{(\hat{x}^a - \alpha^a y^a)}{2} - \frac{\alpha^a ((\hat{x}^a - \alpha^a y^a) - (\hat{x}^b - \alpha^b y^b))}{2(\alpha^a + \alpha^b)} \right] - \left[ \frac{(\hat{x}^b - \alpha^b y^b)}{2} - \frac{\alpha^b ((\hat{x}^b - \alpha^b y^b) - (\hat{x}^a - \alpha^a y^a))}{2(\alpha^a + \alpha^b)} \right] \]
\[ = \left[ \frac{(\hat{x}^a - \alpha^a y^a)}{2} - \frac{(\hat{x}^b - \alpha^b y^b)}{2} \right] - \left[ \frac{(\hat{x}^a - \alpha^a y^a)}{2} - \frac{(\hat{x}^b - \alpha^b y^b)}{2} \right] = 0 \quad \text{(F-36)} \]

Consequently, \( \hat{T_1} - \hat{T_2} = -(\alpha^a + \alpha^b) \xi^b \). \( \square \)
APPENDIX G. PROOF OF PROPOSITION 6

Similar to the approach used for proving Proposition 5, we calculate the regret imposed by adopting policy \( P_2 \) given that each state \( S_1, S_2, \) and \( S_3 \) will have occurred.

**Case A.** Regret imposed by adopting policy \( P_2 \) given that state \( S_1 \) will have occurred, \( \text{Reg}(P_2 | S_1) \).

Regarding the definition of regret, \( \text{Reg}(P_2 | S_1) = \Pi_1^* - \Pi_2 \) where \( \Pi_1^* \) indicates the maximum value of profit that is achievable in the state \( S_1 \) if we know the materialized values of potential demand at the beginning of the pricing process. In addition, \( \Pi_2 \) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \( P_2 \). According to Equation (4), adopting policy \( P_2 \) results in

\[
P^* = \frac{\bar{D}^* + \beta^* q + c^* + \omega^* q - \xi^*}{2} \quad \text{and} \quad p^b = \frac{D^* + \beta^* q + c^b + \omega^* q + \xi^*}{2}.
\]

To calculate the total regret, it is divided into directions \( a \) and \( b \).

\[
\text{Reg}(P_2 | S_1) = \Pi_1^* - \Pi_2 = (\Pi_1^{*a} + \Pi_1^{*b}) - (\Pi_2^{a} + \Pi_2^{b}) = (\Pi_1^{*a} - \Pi_2^{a}) + (\Pi_1^{*b} - \Pi_2^{b}) = R_2^{*a} + R_2^{*b} \quad (G-1)
\]

Since the maximum achievable profit is \( \Pi_1^* \),

\[
R_2^{*a} = \left( \frac{D^* + \beta^* q - \alpha^* (c^* + \omega^* q + \xi^*)}{4 \alpha^*} \right) - (p^* - c^* - \alpha^* q - \xi^*) \frac{\hat{d}_a}{4 \alpha^*} = \left( \frac{\hat{d}_a}{4 \alpha^*} \right)^2 \frac{B_{\alpha}^*}{(p^* - c^* - \alpha^* q - \xi^*) \hat{d}_a} \quad (G-2)
\]

In this case,

\[
\Pi_2^{*a} = (p^* - c^* - \alpha^* q - \xi^*) \frac{\hat{d}_a}{4 \alpha^*} = \left( \frac{D^* + \beta^* q + c^* + \omega^* q - \xi^*}{2} \right) \frac{\hat{d}_a}{4 \alpha^*} = \left( \frac{\hat{d}_a}{4 \alpha^*} \right)^2 \frac{B_{\alpha}^*}{(p^* - c^* - \alpha^* q - \xi^*) \hat{d}_a} \quad (G-3)
\]

and

\[
d^a = \hat{D}^* - \alpha^* \left( \frac{D^* + \beta^* q + c^* + \omega^* q - \xi^*}{2} \right) + \beta^* q = \frac{2 \hat{D}^* - \bar{D}^* + \beta^* q - \alpha^* (c^* + \omega^* q - \xi^*)}{2} = \frac{\Delta^* + \hat{D}^* + \beta^* q - \alpha^* (c^* + \omega^* q + \xi^*)}{2}
\]

\[
= \frac{\Delta^* + \hat{D}^* + \beta^* q - \alpha^* (c^* + \omega^* q + \xi^*)}{2} = \frac{\Delta^* + \hat{D}^* + \beta^* q - \alpha^* (c^* + \omega^* q + \xi^* - \xi^* - \xi^*)}{2} = \frac{\Delta^* + \alpha^* (\xi^* + \xi^* + \hat{T}^*_1)}{2} \quad (G-4)
\]

Hence,
\[
\hat{H}^a_2 = \left( \frac{\hat{D}^b + \beta^a q + c^b + \omega^a q - \xi^a}{2} - (c^b + \omega^a q + \xi^b) \right) \left( \frac{\Delta^a + \alpha^a (\xi^a + \xi^b) + \hat{T}_1}{2} \right)
\]

\[
= \left( \frac{\hat{D}^b - \hat{D}^a + \beta^a q + \alpha^a \left( c^b + \omega^a q + \xi^a - \xi^b \right) - 2\alpha^a (c^b + \omega^a q + \xi^b)}{2\alpha^a} \right) \left( \frac{\Delta^a + \alpha^a (\xi^a + \xi^b) + \hat{T}_1}{2} \right)
\]

\[
= \left( \frac{-\Delta^a + \hat{D}^b + \beta^a q + \alpha^a \left( c^b + \omega^a q + \xi^b \right) - \alpha^a (\xi^a + \xi^b)}{2\alpha^a} \right) \left( \frac{\Delta^a + \alpha^a (\xi^a + \xi^b) + \hat{T}_1}{2} \right)
\]

\[
= \left( \frac{-\Delta^a - \alpha^a (\xi^a + \xi^b) + \hat{T}_1}{2\alpha^a} \right) \left( \Delta^a + \alpha^a (\xi^a + \xi^b) + \hat{T}_1 \right) = \left( \hat{T}_1 \right)^2 - \left( \Delta^a + \alpha^a (\xi^a + \xi^b) \right)^2
\]

(G-5)

and consequently,

\[
R^a_2 = \frac{\left( \hat{T}_1 \right)^2}{4\alpha^a} - \frac{\left( \Delta^a + \alpha^a (\xi^a + \xi^b) \right)^2}{4\alpha^a} = \left( \frac{\Delta^a + \alpha^a (\xi^a + \xi^b)}{4\alpha^a} \right)^2.
\]

(G-6)

Similarly,

\[
R^b_2 = \left( \frac{\hat{D}^b + \beta^b q - \alpha^b (c^b + \omega^b q - \xi^b)}{4\alpha^b} \right)^2 - \frac{u^b_2}{4\alpha^b} \left( \frac{\left( \hat{T}_1 \right)^2}{4\alpha^b} - \left( p^b - c^b - \omega^b q + \xi^b \right) \right) \hat{d}^b
\]

(G-7)

\[
\hat{H}^b_2 = \left( \frac{\hat{D}^b + \beta^b q + c^b + \omega^b q + \xi^a}{2} - (c^b + \omega^a q - \xi^b) \right) \hat{d}^b
\]

(G-8)

and

\[
\hat{d}^b = \hat{D}^b - \alpha^b \left( \frac{\hat{D}^b + \beta^b q + c^b + \omega^b q + \xi^a}{2} \right) + \beta^b q = \frac{2\hat{D}^b - \hat{D}^b + \beta^b q - \alpha^b (c^b + \omega^b q + \xi^b)}{2} = \frac{\Delta^b + \hat{D}^b + \beta^b q - \alpha^b \left( c^b + \omega^b q + \xi^a - \xi^b \right)}{2}
\]

\[
= \frac{\Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2}{2}
\]

(G-9)

Hence,

\[
\hat{H}^b_2 = \left( \frac{\hat{D}^b + \beta^b q + c^b + \omega^b q + \xi^a}{2} - (c^b + \omega^a q - \xi^b) \right) \left( \frac{\Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2}{2} \right)
\]

\[
= \left( \frac{\hat{D}^b - \hat{D}^b + \beta^b q + \alpha^b \left( c^b + \omega^b q + \xi^a - \xi^b \right) - 2\alpha^b (c^b + \omega^a q - \xi^b)}{2\alpha^b} \right) \left( \frac{\Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2}{2} \right)
\]

\[
= \left( \frac{-\Delta^b + \hat{D}^b + \beta^b q + \alpha^b \left( c^b + \omega^b q - \xi^b \right) + \alpha^b (\xi^a + \xi^b)}{2\alpha^b} \right) \left( \frac{\Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2}{2} \right)
\]

\[
= \left( \frac{-\Delta^b + \alpha^b (\xi^a + \xi^b) + \hat{T}_2}{2\alpha^b} \right) \left( \Delta^b - \alpha^b (\xi^a + \xi^b) + \hat{T}_2 \right) = \left( X^b_2 \right)^2 - \left( \Delta^b - \alpha^b (\xi^a + \xi^b) \right)^2
\]

(G-10)
Therefore,
\[
R_{2}^{b} = \frac{\left(\hat{T}_{1}\right)^{2}}{4\alpha^{b}} - \frac{\left(\hat{T}_{2}\right)^{2}}{4\alpha^{b}} - \left(\Delta^{b} - \alpha^{b}\left(\xi^{a} + \xi^{b}\right)\right)^{2} \quad \frac{\left(\Delta^{b} - \alpha^{b}\left(\xi^{a} + \xi^{b}\right)\right)^{2}}{4\alpha^{b}}
\]
and consequently,
\[
\text{Reg}(P_{2} | S_{1}) = R_{2}^{a} + R_{2}^{b} = \frac{\left(\Delta^{a} + \alpha^{a}\left(\xi^{a} + \xi^{b}\right)\right)^{2}}{4\alpha^{a}} + \frac{\left(\Delta^{b} - \alpha^{b}\left(\xi^{a} + \xi^{b}\right)\right)^{2}}{4\alpha^{b}}.
\]  

\textbf{Case B.} Regret imposed by adopting policy } P_{2} \text{ given that state } S_{2} \text{ will have occurred (i.e. } \text{Reg}(P_{2} | S_{2})\).

In this state, \(\text{Reg}(P_{2} | S_{2}) = \Pi_{2}^{*} - \Pi_{2}^{2}\) where \(\Pi_{2}^{2}\) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \(P_{2}\). Similarly, we divide it into two parts, the regret in direction \(a\) and the regret in direction \(b\). Therefore,
\[
\text{Reg}(P_{2} | S_{2}) = \Pi_{2}^{*} - \Pi_{2}^{2} = R_{2}^{2a} + R_{2}^{2b} \tag{G-13}
\]
\[
R_{2}^{2a} = \frac{\left(\hat{D}^{a} + \beta^{a}q - \alpha^{a}\left(c^{a} + \omega^{a}q - \xi^{a}\right)\right)^{2}}{4\alpha^{a}} - \left(p^{a} - c^{a} - \omega^{a}q + \xi^{a}\right)\hat{d}^{a} = \frac{\left(p^{a} - c^{a} - \omega^{a}q + \xi^{a}\right)\hat{d}^{a}}{4\alpha^{a}} + \frac{\left(\hat{T}_{1}\right)^{2}}{4\alpha^{a}} - \left(p^{a} - c^{a} - \omega^{a}q + \xi^{a}\right)\hat{d}^{a}\tag{G-14}
\]
where,
\[
\Pi_{2}^{2a} = \left(p^{a} - c^{a} - \omega^{a}q + \xi^{a}\right)\hat{d}^{a} = \left(\frac{\hat{D}^{a} + \beta^{a}q}{2\alpha^{a}} + \frac{c^{a} + \omega^{a}q - \xi^{a}}{2} - (c^{a} + \omega^{a}q - \xi^{a})\right)\hat{d}^{a}. \tag{G-15}
\]
In addition,
\[
\hat{d}^{a} = \hat{D}^{a} - \alpha^{a}p^{a} + \beta^{a}q = \hat{D}^{a} - \alpha^{a}\left(\frac{\hat{D}^{a} + \beta^{a}q}{2\alpha^{a}} + \frac{c^{a} + \omega^{a}q - \xi^{a}}{2}\right) + \beta^{a}q = \frac{2\hat{D}^{a} - \hat{D}^{a} + \beta^{a}q}{2} - \frac{\alpha^{a}}{2}(c^{a} + \omega^{a}q - \xi^{a})
\]
\[
= \frac{\Delta^{a} + \hat{D}^{a} + \beta^{a}q - \alpha^{a}(c^{a} + \omega^{a}q - \xi^{a})}{2} = \frac{\Delta^{a} + \hat{T}_{1}^{a}}{2}. \tag{G-16}
\]
Hence,
\[
\hat{\Pi}_{2}^{2a} = \frac{\left(\hat{D}^{a} + \beta^{a}q - \alpha^{a}\left(c^{a} + \omega^{a}q - \xi^{a}\right)\right)^{2}}{2\alpha^{a}} - \frac{\left(c^{a} + \omega^{a}q - \xi^{a}\right)^{2}}{2}\left(\frac{\Delta^{a} + \hat{T}_{1}^{a}}{2}\right)
\]
\[
= \frac{\left(\hat{D}^{a} + \beta^{a}q - \alpha^{a}\left(c^{a} + \omega^{a}q - \xi^{a}\right)\right)^{2}}{2\alpha^{a}} - \frac{\left(\Delta^{a} + \hat{T}_{1}^{a}\right)^{2}}{2} = \frac{\left(\hat{T}_{1}^{a}\right)^{2} - \left(\Delta^{a}\right)^{2}}{4\alpha^{a}} \tag{G-17}
\] and
\[ R_2^a = \frac{(\hat{\alpha})^2}{4\alpha^a} - \left[ \frac{(\hat{\alpha})^2 - (\Delta^a)^2}{4\alpha^a} \right] = \frac{(\Delta^a)^2}{4\alpha^a}. \]  
\[ (G-18) \]

Similarly, it could be proven that \( R_2^b = \frac{(\Delta^b)^2}{4\alpha^b} \). Hence,

\[ \text{Reg}(A_i | S_i) = R_2^{2a} + R_2^{2b} = \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b}. \]  
\[ (G-19) \]

**Case C.** Regret imposed by adopting policy \( P_2 \) given that state \( S_3 \) will have occurred (i.e. \( \text{Reg}(P_2 | S_3) \)).

A similar approach can be used in this case. We know that, \( \text{Reg}(P_2 | S_3) = 2\tilde{\Pi}_2 - \tilde{\Pi}_1 \) where \( \tilde{\Pi}_1 \) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \( P_2 \). According to Equation (4), adopting policy \( P_2 \) results in

\[ p^a = \frac{\hat{x}^a + \beta^a q + \frac{\Delta^a}{2} + \frac{\Delta^b}{2} - \frac{\Delta^a}{2} - \frac{\Delta^b}{2}}{2\alpha^a} = \frac{x^a + y^a - \xi^a}{2} \]  
\[ (G-20) \]

and

\[ p^b = \frac{\hat{x}^b + \beta^b q + \frac{\Delta^b}{2} + \frac{\Delta^a}{2} - \frac{\Delta^b}{2} - \frac{\Delta^a}{2}}{2\alpha^b} = \frac{x^b + y^b + \xi^a}{2} \]  
\[ (G-21) \]

As mentioned, the maximum achievable profit in state \( S_3 \) is

\[ \tilde{\Pi}_2 = \frac{\left( \alpha^a \hat{x}^a + \alpha^b \hat{x}^b - \alpha^a \alpha^b \left( y^a + y^b \right) \right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \frac{\left( \alpha^a \hat{x}^a + \alpha^b \hat{x}^b - \alpha^a \alpha^b \left( y^a + y^b + \xi^a + \xi^b \right) \right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} \]

\[ = \frac{\left( \alpha^a \hat{x}^a - \alpha^b \left( y^a + \xi^a \right) + \alpha^b \left( \hat{x}^b - \alpha^a \left( y^b + \xi^b \right) \right) \right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} = \frac{\left( \alpha^a \hat{x}^a + \alpha^b \hat{x}^b \right)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} \]  
\[ (G-22) \]

where \( \hat{x}^a = \hat{D}^a + \beta^a q, \; \hat{x}^b = \hat{D}^b + \beta^b q, \; \hat{\xi}^a = \hat{x}^a - \alpha^a \left( y^a - \xi^a \right), \) and \( \hat{\xi}^b = \hat{x}^b - \alpha^b \left( y^b + \xi^a \right) \).

To calculate the total actual profit (\( \tilde{\Pi}_2 \)), we apportion it into two segments, the profit in direction \( a \) and the profit in direction \( b \).

\[ \tilde{\Pi}_2 = \tilde{\Pi}_2^{2a} + \tilde{\Pi}_2^{2b} = \left( p^a - y^a \right) \hat{d}^a + \left( p^b - y^b \right) \hat{d}^b \]  
\[ (G-23) \]

In this case,
\[
\hat{d}^a = \hat{D}^a - \alpha^a\left(\frac{x^a + y^a - \xi^a}{2}\right) + \beta^a q = \hat{x}^a - \alpha^a\left(\frac{y^a - \xi^a}{2}\right) = \frac{2\hat{x}^a - x^a - \alpha^a\left(y^a - \xi^a\right)}{2}
\]
\[
\hat{d}^b = \hat{D}^b - \alpha^b\left(\frac{x^b + y^b + \xi^a}{2}\right) + \beta^b q = \hat{x}^b - \alpha^b\left(\frac{y^b + \xi^a}{2}\right) = \frac{2\hat{x}^b - x^b - \alpha^b\left(y^b + \xi^a\right)}{2}
\]
\[
\hat{\theta}^a = \frac{\Delta^a + \hat{\theta}^a}{2} = \frac{\Delta^a + \hat{\theta}^a}{2} - \alpha^a\left(y^a - \xi^a\right) = \frac{\Delta^a + \hat{\theta}^a}{2}
\]
\[
\hat{\theta}^b = \frac{\Delta^b + \hat{\theta}^b}{2} = \frac{\Delta^b + \hat{\theta}^b}{2} + \alpha^b\left(y^b + \xi^a\right)
\]
\[
\hat{\eta}^a = \frac{\Delta^a + \hat{\eta}^a}{2} = \frac{\Delta^a + \hat{\eta}^a}{2} - \xi^a\left(\Delta^a + \hat{\eta}^a\right) = \frac{\Delta^a + \hat{\eta}^a}{2}
\]
\[
\hat{\eta}^b = \frac{\Delta^b + \hat{\eta}^b}{2} = \frac{\Delta^b + \hat{\eta}^b}{2} + \xi^a\left(\Delta^b + \hat{\eta}^b\right)
\]
\[
\hat{\eta}^a = \left(p^a - c^a - \omega^a q\right)\hat{d}^a = \left(\frac{x^a + y^a - \xi^a}{2}\right)\left(\Delta^a + \hat{\eta}^a\right) = \frac{\Delta^a + \hat{\eta}^a}{2}
\]
\[
\hat{\eta}^b = \left(p^b - c^b - \omega^b q\right)\hat{d}^b = \left(\frac{x^b + y^b + \xi^a}{2}\right)\left(\Delta^b + \hat{\eta}^b\right) = \frac{\Delta^b + \hat{\eta}^b}{2}
\]

Hence,
\[
\hat{\eta}^a = \left(p^a - c^a - \omega^a q\right)\hat{d}^a = \left(\frac{x^a + y^a - \xi^a}{2}\right)\left(\Delta^a + \hat{\eta}^a\right) = \frac{\Delta^a + \hat{\eta}^a}{2}
\]
\[
\hat{\eta}^b = \left(p^b - c^b - \omega^b q\right)\hat{d}^b = \left(\frac{x^b + y^b + \xi^a}{2}\right)\left(\Delta^b + \hat{\eta}^b\right) = \frac{\Delta^b + \hat{\eta}^b}{2}
\]

Similarly,
\[
\hat{\eta}^a = \left(p^a - c^a - \omega^a q\right)\hat{d}^a = \left(\frac{x^a + y^a - \xi^a}{2}\right)\left(\Delta^a + \hat{\eta}^a\right) = \frac{\Delta^a + \hat{\eta}^a}{2}
\]
\[
\hat{\eta}^b = \left(p^b - c^b - \omega^b q\right)\hat{d}^b = \left(\frac{x^b + y^b + \xi^a}{2}\right)\left(\Delta^b + \hat{\eta}^b\right) = \frac{\Delta^b + \hat{\eta}^b}{2}
\]

Now, according to Equations (G-22), (G-26) and (G-29), the corresponding regret can be calculated as follows.
\[ \Pi_1 - \hat{\Pi}_1 = \frac{(\alpha^\ast \hat{T}_2 + \alpha^b \hat{T}_1)}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \frac{(\Delta^\ast)}{4\alpha^a} - \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \hat{T}_2 - \Delta^a - \hat{T}_1)}{2} \right) \]

\[ \frac{(\alpha^\ast \hat{T}_2 + \alpha^b \hat{T}_1)}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \frac{(\alpha^\ast \hat{T}_2)}{4\alpha^a} - \frac{(\alpha^b \hat{T}_1)}{4\alpha^b} + \frac{(\Delta^a)}{4\alpha^a} - \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \hat{T}_2 - \Delta^a - \hat{T}_1)}{2} \right) \]

\[ \frac{(\alpha^\ast \hat{T}_2)}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \frac{(\alpha^\ast \hat{T}_2)}{4\alpha^a} - \frac{(\alpha^b \hat{T}_1)}{4\alpha^b} + \frac{(\Delta^a)}{4\alpha^a} - \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \hat{T}_2 - \Delta^a - \hat{T}_1)}{2} \right) \]

\[ \frac{\xi^a \left( \frac{(\Delta^a + \hat{T}_2 - \Delta^a - \hat{T}_1)}{2} \right)}{2} \]

Regarding the **Lemma 2**,

\[ \Pi_2 - \hat{\Pi}_2 = \frac{(\alpha^\ast + \alpha^b) (\xi^a - \xi^b)}{4(\alpha^a + \alpha^b)} + \frac{(\Delta^a)}{4\alpha^a} + \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \Delta^b)}{2} \right) \]

\[ \frac{(\Delta^a)}{4\alpha^a} + \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \Delta^b)}{2} \right) \]

\[ \frac{(\Delta^a)}{4\alpha^a} + \frac{(\Delta^b)}{4\alpha^b} + \xi^a \left( \frac{(\Delta^a + \Delta^b)}{2} \right) \]

\[ (G-30) \]

**LEMMA 2.** \( \hat{T}_1 - \hat{T}_2 = (\alpha^a + \alpha^b) \xi^a \).

**PROOF.**

\[ \hat{T}_1 - \hat{T}_2 = (\hat{x}^a - \alpha^a (y^a - \xi^a)) - (\hat{x}^b - \alpha^b (y^b + \xi^a)) \]

\[ = \frac{(\hat{x}^a - \alpha^a (y^a - \xi^a))}{K_1} - \frac{(\hat{x}^b - \alpha^b (y^b + \xi^a))}{K_1} \]

\[ = (\hat{T}^a + \beta^a q - \alpha^a y^a) - (\hat{T}^b + \beta^b q - \alpha^b y^b) + (\alpha^a + \alpha^b) \xi^a \]  

\[ (G-32) \]

According to **Lemma 1**, \( K_1 - K_2 = 0 \). Hence, \( \hat{T}_1 - \hat{T}_2 = (\alpha^a + \alpha^b) \xi^a \). □
APPENDIX H. PROOF OF PROPOSITION 7

A similar approach can be used to prove this proposition in three cases.

Case A. Regret imposed by adopting policy $P_3$ given that state $S_1$ will have occurred (i.e. $\text{Reg}(P_3 | S_1)$)

In this case, $\text{Reg}(P_3 | S_1) = \hat{H}_1 - \tilde{H}_1$ where $\hat{H}_1$ demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy $P_3$. According to Equation (4), adopting policy $P_3$ results in

$$ p^o = \frac{\hat{D}^{o} + \hat{P}^{o} q}{2\alpha^o} + \frac{(\hat{D}^{b} + \beta^b q - (\hat{D}^{b} + \beta^b q) + \alpha^b (c^o + \omega^o q + c^b + \omega^b q)}{2(\alpha^o + \alpha^b)} = \frac{x^o - x^b + \alpha^b (y^o + y^b)}{2(\alpha^o + \alpha^b)} \quad (H-1) $$

and

$$ p^b = \frac{\hat{D}^{b} + \beta^b q}{2\alpha^b} + \frac{(\hat{D}^{b} + \beta^b q - (\hat{D}^{b} + \beta^b q) + \alpha^o (c^o + \omega^o q + c^b + \omega^b q)}{2(\alpha^o + \alpha^b)} = \frac{x^b - x^o + \alpha^o (y^o + y^b)}{2(\alpha^o + \alpha^b)} \quad (H-2) $$

According to Table-1, the maximum achievable profit is

$$ \hat{H}_1 = \left( \frac{\hat{D}^{o} + \beta^o \hat{q} - \alpha^o (c^o + \omega^o \hat{q} + \hat{\xi}^o)}{4\alpha^o} \right) \hat{a} + \left( \frac{\hat{D}^{b} + \beta^b \hat{q} - \alpha^b (c^b + \omega^b \hat{q} - \hat{\xi}^b)}{4\alpha^b} \right) \hat{a} = \hat{H}_1^{o} + \hat{H}_1^{b}. \quad (H-3) $$

In addition,

$$ \hat{H}_1 = \left( p^o - (y^o + \hat{\xi}^o) \right) \hat{a} + \left( p^b - (y^b - \hat{\xi}^b) \right) \hat{a} = \hat{H}_1^{o} + \hat{H}_1^{b}. \quad (H-4) $$

In this case,

$$ \hat{a}^o = \hat{D}^{o} - \alpha^o \left( \frac{x^o + x^b - x^o + \alpha^b (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right) + \beta^o \hat{q} = \frac{2\hat{D}^{o} - x^o + 2\beta^o \hat{q} - \alpha^o \left( \frac{x^o - x^b + \alpha^b (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right)}{2(\alpha^o + \alpha^b)} $$

and

$$ \hat{a}^b = \hat{D}^{b} - \alpha^b \left( \frac{x^b + x^o - x^b + \alpha^o (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right) + \beta^b \hat{q} = \frac{2\hat{D}^{b} - x^b + 2\beta^b \hat{q} - \alpha^b \left( \frac{x^b - x^o + \alpha^o (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right)}{2(\alpha^o + \alpha^b)} $$

as well as

$$ \hat{a}^o = \hat{D}^{o} - \alpha^o \left( \frac{x^o + x^b - x^o + \alpha^b (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right) + \beta^o \hat{q} = \frac{2\hat{D}^{o} - x^o + 2\beta^o \hat{q} - \alpha^o \left( \frac{x^o - x^b + \alpha^b (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right)}{2(\alpha^o + \alpha^b)} $$

and

$$ \hat{a}^b = \hat{D}^{b} - \alpha^b \left( \frac{x^b + x^o - x^b + \alpha^o (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right) + \beta^b \hat{q} = \frac{2\hat{D}^{b} - x^b + 2\beta^b \hat{q} - \alpha^b \left( \frac{x^b - x^o + \alpha^o (y^o + y^b)}{2(\alpha^o + \alpha^b)} \right)}{2(\alpha^o + \alpha^b)} $$
\[
\frac{\Delta^a}{2(\alpha^a + \alpha^b)} + \frac{\alpha^b\left( x^a - \alpha^a \left( y^a + \xi^b \right) \right) + \alpha^c\left( x^b - \alpha^b \left( y^b + \xi^a \right) \right)}{2(\alpha^a + \alpha^b)} = \Delta^b + \frac{\alpha^b T_i + \alpha^c T_j}{2(\alpha^a + \alpha^b)} = \Delta^b + \frac{\tau_i}{2} - \frac{\tau_j}{2(\alpha^a + \alpha^b)}.
\]

Therefore,
\[
\hat{\Pi}_3^{*} = \left( p^a - \left( y^a + \xi^b \right) \right) \hat{d}^a = \left( \frac{x^a}{2\alpha^a} + \frac{x^b - \alpha^b \left( y^b + \xi^a \right)}{2(\alpha^a + \alpha^b)} \right) - \left( \frac{x^b}{\alpha^b} + \frac{x^a - \alpha^a \left( y^a + \xi^b \right)}{2(\alpha^a + \alpha^b)} \right)
\]
\[
\Delta^a + \frac{\alpha^b T_i + \alpha^c T_j}{2(\alpha^a + \alpha^b)} = \Delta^b + \frac{\tau_i}{2} - \frac{\tau_j}{2(\alpha^a + \alpha^b)}.
\]

We know that \( T_i = -\Delta^a + \tilde{T}_i \) and \( T_j = -\Delta^b + \tilde{T}_j \). Consequently,
\[
\hat{\Pi}_i^{*} = \left( \frac{T_i}{2\alpha^a} + \frac{T_j}{2(\alpha^a + \alpha^b)} \right) \left( \Delta^a + \frac{T_i}{2} - \alpha^a \left( T_i - T_j \right) \right)
\]
\[
= \frac{\hat{T}_i^{*} - \left( \Delta^a \right)^2}{4\alpha^a} + \frac{2\alpha^a \left( T_i - T_j \right)}{4(\alpha^a + \alpha^b)} \alpha^a \left( T_i - T_j \right)^2
\]

and
\[
\hat{\Pi}_j^{*} = \left( \frac{T_j}{2\alpha^b} + \frac{T_j}{2(\alpha^a + \alpha^b)} \right) \left( \Delta^b + \frac{T_j}{2} - \alpha^b \left( T_i - T_j \right) \right)
\]
\[
= \frac{\hat{T}_j^{*} - \left( \Delta^b \right)^2}{4\alpha^b} + \frac{2\alpha^b \left( T_i - T_j \right)}{4(\alpha^a + \alpha^b)} \alpha^b \left( T_i - T_j \right)^2
\]

According to Lemma 1,
\[
\hat{\Pi}_i^{*} = \frac{\hat{T}_i^{*} - \left( \Delta^a \right)^2}{4\alpha^a} + \frac{2\alpha^a \left( \alpha^a + \alpha^b \right) }{4(\alpha^a + \alpha^b)} \alpha^a \left( \alpha^a + \alpha^b \right) \left( \Delta^a \right)^2
\]
\[
= \frac{\hat{T}_i^{*} - \left( \Delta^a + \alpha^b \xi^b \right)^2}{4\alpha^a}
\]

and,
\[
\hat{\Pi}_j^{*} = \frac{\hat{T}_j^{*} - \left( \Delta^b \right)^2 - \alpha^b \left( \alpha^a + \alpha^b \right) \left( \Delta^a \right)^2}{4\alpha^b}
\]
\[
= \frac{\hat{T}_j^{*} - \left( \Delta^a - \alpha^b \xi^b \right)^2}{4\alpha^b}
\]

Hence,
\[
\text{Reg}(P_3 | S_1) = \Pi_3^{*} - \hat{\Pi}_3^{*} = \left( \frac{\hat{T}_i^{*}}{4\alpha^a} + \frac{\hat{T}_j^{*}}{4\alpha^b} \right) - \left( \frac{\hat{T}_i^{*}}{4\alpha^a} - \frac{\left( \Delta^a + \alpha^a \xi^b \right)^2}{4\alpha^a} \right) - \left( \frac{\hat{T}_j^{*}}{4\alpha^b} - \frac{\left( \Delta^a - \alpha^b \xi^b \right)^2}{4\alpha^b} \right)
\]

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\[
\frac{(\Delta^a + \alpha^a \xi^a)^2}{4\alpha^a} + \frac{(\Delta^b - \alpha^b \xi^b)^2}{4\alpha^b}
\]  

(H-12)

**Case B.** Regret imposed by adopting policy \( P_3 \) given that state \( S_2 \) will have occurred (i.e. \( \text{Reg}(P_3 | S_2) \))

In this case, \( \text{Reg}(P_3 | S_2) = \Pi_3^* - \Pi_3^f \) where \( \Pi_3^f \) demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy \( P_3 \). According to Table-1, the maximum achievable profit is

\[
\Pi^*_3 = \left( \frac{\hat{D}^* + \beta^* q - \alpha^* \left( c^* + \alpha^* q - \xi^* \right)}{4\alpha^*} \right)^2 + \left( \frac{\hat{D}^b + \beta^b q - \alpha^b \left( c^b + \alpha^b q + \xi^b \right)}{4\alpha^b} \right)^2 = \frac{(\hat{\Pi}_3^*)^2}{4\alpha^*} + \frac{(\hat{\Pi}_3^f)^2}{4\alpha^b}. 
\]  

(H-13)

In addition,

\[
\Pi_3^f = (p^* - (y^* - \xi^*)) \hat{d}^a + (p^b - (y^b + \xi^b)) \hat{d}^b = \Pi_3^{a*} + \Pi_3^{b*}. 
\]  

(H-14)

Because,

\[
\hat{d}^a = \hat{D}^a - \alpha^a \left( \frac{x^a + x^b + \alpha^b (y^* + y^b)}{2(\alpha^a + \alpha^b)} \right) + \beta^a q = \frac{2\hat{D}^a - x^* + \beta^a q - \alpha^a \left( \frac{x^a - x^b + \alpha^b (y^* + y^b)}{2(\alpha^a + \alpha^b)} \right)}{2} \\
= \frac{x^*}{2} - \alpha^a \left( \frac{x^a - x^b + \alpha^b (y^* + y^b)}{2(\alpha^a + \alpha^b)} \right) = \Delta^a + \frac{x^*}{2} - \alpha^a \left( \frac{x^a - x^b + \alpha^b (y^* - \xi^* + y^b + \xi^b)}{2(\alpha^a + \alpha^b)} \right) \\
= \Delta^a + \frac{\alpha^b (x^a - \alpha^a (y^* - \xi^*)) + \alpha^a (x^b - \alpha^b (y^* + \xi^b))}{2(\alpha^a + \alpha^b)} = \Delta^a + \frac{\alpha^b T_1^* + \alpha^a T_2^*}{2(\alpha^a + \alpha^b)} = \Delta^a + \left[ \frac{T_1^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right], 
\]  

(H-15)

and

\[
\hat{d}^b = \Delta^b + \frac{\alpha^b T_1^* + \alpha^a T_2^*}{2(\alpha^a + \alpha^b)} = \Delta^b + \left[ \frac{T_2^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right]. 
\]  

(H-16)

We can calculate \( \Pi_3^{a*} \) and \( \Pi_3^{b*} \) as follows:

\[
\Pi_3^{a*} = (p^* - (y^* - \xi^*)) \hat{d}^a = \left( \frac{x^a}{2\alpha^*} + \frac{x^a - x^b + \alpha^b (y^* + y^b)}{2(\alpha^a + \alpha^b)} \right) (y^* - \xi^*) \left( \Delta^a + \left[ \frac{T_2^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right] \right) \\
= \left( \frac{x^a}{2\alpha^*} + \frac{x^a - x^b + \alpha^b (y^* - \xi^* + y^b + \xi^b)}{2(\alpha^a + \alpha^b)} \right) (y^* + \xi^b) \left( \Delta^a + \left[ \frac{T_2^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right] \right) \\
= \left( \frac{2\alpha^a + \alpha^b}{2(\alpha^a + \alpha^b)} \right) \left( \Delta^a + \frac{T_2^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right) \left( \Delta^a + \left[ \frac{T_2^*}{2} - \frac{\alpha^a (T_1^* - T_2^*)}{2(\alpha^a + \alpha^b)} \right] \right). 
\]  

(H-17)
We also know that $T_i^* = -\Delta^a + \hat{T}_i^*$, and similarly $T_j^* = -\Delta^b + \hat{T}_j^*$. Consequently,

$$\hat{H}_{3a} = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\Delta^a + \frac{T_i^*}{\alpha^a + \alpha^b}\right) = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right)$$

and

$$\hat{H}_{3b} = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\Delta^b + \frac{T_j^*}{\alpha^a + \alpha^b}\right) = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right).$$

Furthermore, according to Lemma 2,

$$\hat{H}_{3a} = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\Delta^a + \frac{T_i^*}{\alpha^a + \alpha^b}\right) = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right)$$

and

$$\hat{H}_{3b} = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\Delta^b + \frac{T_j^*}{\alpha^a + \alpha^b}\right) = \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right) \left(\frac{T_j^* - T_i^*}{2\alpha^a} + \frac{T_i^* - T_j^*}{2\alpha^a + \alpha^b}\right).$$

Hence,

$$\text{Reg}(P_3 | S_3) = \Pi_3^* - \Pi_3^* = \left(\frac{T_i^*}{2\alpha^a} + \frac{T_j^*}{2\alpha^a + \alpha^b}\right) \left(\Delta^a - \frac{T_i^*}{\alpha^a + \alpha^b}\right) \left(\Delta^b + \frac{T_j^*}{\alpha^a + \alpha^b}\right)$$

$$(H-22)$$

Case C. Regret imposed by adopting policy $P_3$ given that state $S_3$ will have occurred (i.e. $\text{Reg}(P_3 | S_3)$)

According to the regret definition, $\text{Reg}(P_3 | S_3) = \Pi_3^* - \hat{\Pi}_3^*$ where $\hat{\Pi}_3^*$ demonstrates the total actual profit obtained at the end of the planning horizon regarding the prices determined by policy $P_3$. According to Equation (F-24), the maximum achievable profit is

$$\Pi_3^* = \left(\alpha^a \hat{T}_3 + \alpha^b \hat{T}_3\right)^2 / 4\alpha^a \alpha^b \left(\alpha^a + \alpha^b\right).$$

$$(H-23)$$

To calculate the total actual profit, we should note that the company would have empty repositioning costs despite this policy’s attempt to equalize demands in both directions. In other words, the actual value of the
realized demands (i.e. \(\hat{d}^a\) and \(\hat{d}^b\)) would differ from the planned values (i.e. \(d^a\) and \(d^b\)) due to the high possibility of change in the value of potential demand (i.e. \(\hat{\Delta}^a\) and \(\hat{\Delta}^b\)) to \(\Delta^a\) and \(\Delta^b\) respectively. Therefore, adopting this policy would result in empty repositioning in direction \(b\) if \(\Delta^b \geq \Delta^a\) and \(a\) if \(\Delta^a \leq \Delta^b\) respectively.

Regarding this variability, we will calculate the corresponding regret in two cases.

**Case C-1.** if \(\Delta^a \geq \Delta^b\) (called case \(A^+\)),

In this case, \(\hat{\Pi}_3 = \left(p^a - (y^a + \tilde{\varepsilon}^a)\right)\hat{d}^a + \left(p^b - (y^b - \tilde{\varepsilon}^b)\right)\hat{d}^b = \hat{\Pi}_3^{\text{a}} + \hat{\Pi}_3^{\text{b}}\). In addition, according to Equations (H-7) and (H-10),

\[
\hat{\Pi}_3^{\text{a}} = \left(p^a - (y^a + \tilde{\varepsilon}^a)\right)\hat{d}^a = \left(\frac{T_r}{2\alpha^a} + \frac{T_2 - T_1}{2(\alpha^a + \alpha^b)}\right)\hat{d}^a = \left(\frac{\hat{T}_1 - (\Delta^a + \alpha^a \tilde{\varepsilon}^a)}{2\alpha^a}\right)\hat{d}^a = \left(\frac{\hat{T}_1}{4\alpha^a}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2
\]

and according to Equation (H-11),

\[
\hat{\Pi}_3^{\text{b}} = \left(p^b - (y^b - \tilde{\varepsilon}^b)\right)\hat{d}^b = \left(\frac{T_2}{2\alpha^b} + \frac{T_2 - T_1}{2(\alpha^a + \alpha^b)}\right)\hat{d}^b = \left(\frac{\hat{T}_2 - (\Delta^b - \alpha^b \tilde{\varepsilon}^b)}{2\alpha^b}\right)\hat{d}^b = \left(\frac{\hat{T}_2}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

Hence, the actual regret of policy \(P_3\) in this condition is

\[
\Pi_3 - \hat{\Pi}_3 = \left(\frac{\alpha^a \hat{T}_3 + \alpha^b \hat{T}_3}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)}\right)^2 - \left(\frac{\hat{T}_1}{4\alpha^a}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 - \left(\frac{\hat{T}_2}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2 = \left(\frac{\alpha^a \hat{T}_3 + \alpha^b \hat{T}_3}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)}\right)^2 - \left(\frac{\hat{T}_1}{4\alpha^a}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 - \left(\frac{\hat{T}_2}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

Using the result of **Lemma 1** provides the following result

\[
\Pi_3 - \hat{\Pi}_3 = \left(-\left(\alpha^a + \alpha^b\right) \tilde{\varepsilon}^a\right)^2 + \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2 = \left(\frac{\alpha^a + \alpha^b}{4\alpha^a + \alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^a + \alpha^b}{4\alpha^a + \alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a + \alpha^b}{4\alpha^a + \alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^a + \alpha^b}{4\alpha^a + \alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]

\[
= \left(\frac{\alpha^a}{4\alpha^a} + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^a + \alpha^a \tilde{\varepsilon}^a\right)^2 + \left(\frac{\alpha^b}{4\alpha^b}\right)^2 \left(\Delta^b - \alpha^b \tilde{\varepsilon}^b\right)^2
\]
Case C-2. if $\Delta^a < \Delta^b$ (called case $A^b$),

In this case, $\tilde{\Pi}^a = (p^a - (y^a - \xi^a))d^a + (p^b - (y^b + \xi^a))d^b = \tilde{\Pi}^{a\cdot}_a + \tilde{\Pi}^{b\cdot}_b$. In addition, according to Equations (H-17) and (H-20),

$$\tilde{\Pi}^{a\cdot}_a = (p^a - (y^a - \xi^a))d^a = \left( \frac{T'_a}{2\alpha^a} + \frac{T'_a - T'_b}{2(\alpha^a + \alpha^b)} \right)d^a = \left( \frac{\tilde{T}'_a - (\Delta^a - \alpha^a \xi^a)}{2\alpha^a} \right) \left( \frac{\tilde{T}'_a + (\Delta^a - \alpha^a \xi^a)}{2} \right)$$

and according to Equation (H-21),

$$\tilde{\Pi}^{b\cdot}_b = (p^b - (y^b + \xi^a))d^b = \left( \frac{T'_b}{2\alpha^b} + \frac{T'_b - T'_a}{2(\alpha^a + \alpha^b)} \right)d^b = \left( \frac{\tilde{T}'_b - (\Delta^b + \alpha^b \xi^a)}{2\alpha^b} \right) \left( \frac{\tilde{T}'_b + (\Delta^b + \alpha^b \xi^a)}{2} \right)$$

Hence, the actual regret of policy $P_3$ in this condition is

$$\Pi^3 - \tilde{\Pi}^3 = \frac{(\alpha^a \tilde{T}'_a + \alpha^b \tilde{T}'_b)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)} - \frac{(\tilde{T}'_a - (\Delta^a - \alpha^a \xi^a)^2}{4\alpha^a} - \frac{(\tilde{T}'_b - (\Delta^b + \alpha^b \xi^a)^2}{4\alpha^b} - \frac{(\alpha^a \tilde{T}'_a + \alpha^b \tilde{T}'_b)^2}{4\alpha^a \alpha^b (\alpha^a + \alpha^b)}$$

Using the result of Lemma 2 provides the following result

$$\Pi^3 - \tilde{\Pi}^3 = -\frac{(\alpha^a + \alpha^b) \xi^a}{4(\alpha^a + \alpha^b)} + \frac{(\Delta^a - \alpha^a \xi^a)^2}{4\alpha^a} + \frac{(\Delta^b + \alpha^b \xi^a)^2}{4\alpha^b}$$

$$= \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\xi^a}{2}(\Delta^b - \Delta^a)$$

Since $P(\Delta^a \geq \Delta^b) = p'_1$ and $P(\Delta^a < \Delta^b) = 1 - p'_1 = p'_2$, the expected regret in this case is calculated as follows:

$$\text{Reg}(A_1 | S_2) = \text{Reg}(A_1 | S_2 \cap A^*) p(A^*) + \text{Reg}(A_1 | S_2 \cap A') p(A')$$

$$= \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\xi^a}{2}(\Delta^b - \Delta^a) \right) p'_1 + \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\xi^a}{2}(\Delta^b - \Delta^a) \right) p'_2$$
\[
\frac{\alpha^2}{4\alpha^2} + \frac{\beta^2}{4\alpha^2} \left( p_1 \frac{\alpha^2}{2} (\Delta^a - \Delta^b) + p_2 \frac{\alpha^2}{2} (\Delta^b - \Delta^a) \right)
\]

(H-32)

It is worthwhile to note that the value of \( p_1 \) can be calculated as follows:

\[
p_1 = P(\Delta^a - \Delta^b \geq 0) = P\left( (\hat{\Delta}^a - \hat{\Delta}^b) - (\hat{\Delta}^b - \hat{\Delta}^a) \geq 0 \right) = P\left( (\hat{\Delta}^a - \hat{\Delta}^b) \geq (\hat{\Delta}^b - \hat{\Delta}^a) \right) = P\left( (\hat{\Delta}^a - \hat{\Delta}^b) \geq D^a + \frac{\Delta^a}{2} R^a - D^b + \frac{\Delta^b}{2} R^b \right) = P\left( (\hat{\Delta}^a - \hat{\Delta}^b) \geq D^a - D^b + \frac{\Delta^a}{2} (R^a - R^b) \right)
\]

(H-33)

Now, utilizing Proposition 2 concludes that

\[
p'_1 = \begin{cases} 
0 & D^a - D^b \leq \Gamma \\
\frac{\left( (R^a - \frac{\Delta^a}{2} (R^a - R^b)) \right)^2}{2 R^a R^b} & D^a - D^b \leq \Gamma \& D^a - D^b \leq \Gamma \\
\frac{R^a + (1 - \frac{\Delta^a}{2}) (R^a - R^b)}{2 R^a} & D^a - D^b \geq \Gamma \& D^a - D^b \leq \Gamma \\
\frac{R^a - \frac{\Delta^a}{2} (R^a - R^b)}{2 R^a} & D^a - D^b \leq \Gamma \& D^a - D^b \geq \Gamma \\
1 - \frac{\left( (R^a + \frac{\Delta^a}{2} (R^a - R^b)) \right)^2}{2 R^a R^b} & D^a - D^b \geq \Gamma \& D^a - D^b \geq \Gamma \\
1 & D^a - D^b \geq \Gamma
\end{cases}
\]

(H-24)

where \( \Gamma = D^a - D^b + \frac{\Delta^a}{2} (R^a - R^b) \). □
APPENDIX I. PROOF OF PROPOSITION 8

To determine the optimal policy, we will first calculate the expected regret of any possible policies as follows:

\[
E^R(p_1) = E^R(p_1|s_1)p_1 + E^R(p_1|s_2)p_2 + E^R(p_1|s_3)p_3 = \\
= p_1 \left( \frac{(\Delta^a)^2}{4\alpha^a} + \left(\frac{\Delta^b}{4\alpha^b}\right)^2 \right) + p_2 \left( \frac{(\Delta^a - \alpha^a\xi^a + \xi^b)^2}{4\alpha^a} + \frac{(\Delta^b + \alpha^b\xi^a + \xi^b)^2}{4\alpha^b} \right) \\
+ p_3 \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
= \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
(I-1)
\]

Also,

\[
E^R(p_2) = E^R(p_2|s_1)p_1 + E^R(p_2|s_2)p_2 + E^R(p_2|s_3)p_3 = \\
= p_1 \left( \frac{(\Delta^a)^2}{4\alpha^a} + \left(\frac{\Delta^b}{4\alpha^b}\right)^2 \right) + p_2 \left( \frac{(\Delta^a - \alpha^a\xi^a + \xi^b)^2}{4\alpha^a} + \frac{(\Delta^b + \alpha^b\xi^a + \xi^b)^2}{4\alpha^b} \right) \\
+ p_3 \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
= \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
(I-2)
\]

Finally,

\[
E^R(p_3) = E^R(p_3|s_1)p_1 + E^R(p_3|s_2)p_2 + E^R(p_3|s_3)p_3 = \\
= p_1 \left( \frac{(\Delta^a + \alpha^a\xi^a)^2}{4\alpha^a} + \left(\frac{\Delta^b - \alpha^b\xi^a}{4\alpha^b}\right)^2 \right) + p_2 \left( \frac{(\Delta^a - \alpha^a\xi^a)^2}{4\alpha^a} + \frac{(\Delta^b + \alpha^b\xi^a)^2}{4\alpha^b} \right) \\
+ p_3 \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
= \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} + \frac{\alpha^a + \alpha^b}{4} \left( \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^a)^2}{4\alpha^a} + \frac{(\Delta^b)^2}{4\alpha^b} \right) \\
+ p_3 \left( \frac{p_1^2}{2} \xi^a (\Delta^a - \Delta^b) + \frac{p_2}{2} \xi^a (\Delta^b - \Delta^a) \right) \\
(I-3)
\]

Removing term \(\frac{(\Delta^a)^2}{4\alpha^a}\) from Equations (I-1), (I-2), and (I-3) does not change the policies’ preference due to repetition. Therefore, for the sake of simplicity, we remove this term from the calculated expected
values and call the remaining term the expected additional regret. The value of expected additional regret \((E^{AR})\) can be determined by utilizing **Lemma 3**, 

\[
E^{AR}(P_2) = P_1\left( \frac{(1-\lambda)(R^a - R^b)\left(\xi^a + \xi^b\right)}{4} + \left(\alpha^a + \alpha^b\right)\left(\xi^a + \xi^b\right) \right) + P_3\left( \frac{(\alpha^a + \alpha^b)(\xi^a)^2 + (1-\lambda)\xi^b\left(R^a - R^b\right)}{4} \right)
\]

\[
= P_1\left( \frac{(\xi^a + \xi^b)}{4} \right)\left(1-\lambda\right)(R^a - R^b) + \left(\alpha^a + \alpha^b\right)\left(\xi^a + \xi^b\right)\right) + P_3\left( \frac{(\alpha^a + \alpha^b)(\xi^a)^2 + (1-\lambda)\xi^b\left(R^a - R^b\right)}{4} \right).
\]

In addition, utilizing the result of **Lemma 3** in Equation (I-2) results in

\[
E^{AR}(P_2) = P_1\left( \frac{(\xi^a + \xi^b)}{4} \right)\left(1-\lambda\right)(R^a - R^b) + \left(\alpha^a + \alpha^b\right)\left(\xi^a + \xi^b\right)\right) + P_3\left( \frac{(\alpha^a + \alpha^b)(\xi^a)^2 + (1-\lambda)\xi^b\left(R^a - R^b\right)}{4} \right).
\]

Finally,

\[
E^{AR}(P_3) = (1-\lambda)(R^a - R^b)\left( \frac{p_1 + p_3p'_1\xi^a - (p_2 + p_3p'_1)\xi^b}{4} \right) + \left(\alpha^a + \alpha^b\right)\left(\frac{p_1\xi^a}{4} + \frac{p_2\xi^b}{4} \right).
\]

Regarding the obtained results, the optimal policy is the one resulting in the least value of average additional regret \((E^{AR})\).□

**LEMMA 3.** \(E\left(\Delta^a - \Delta^b\right) = \left(\frac{1-\lambda}{2}\right)(R^a - R^b)\).

**PROOF.**

\[
E\left(\Delta^a - \Delta^b\right) = E\left(\hat{D}^a - \hat{D}^b\right) - E\left(\hat{D}^b - \hat{D}^a\right) = E\left(D^a + \frac{\lambda}{2}R^a - D^b - \frac{\lambda}{2}R^b\right) - E\left(D^b + \frac{\lambda}{2}R^b - D^a - \frac{\lambda}{2}R^a\right)
\]

\[
= \frac{D^a - D^b}{2} - \frac{D^b - D^a}{2} - \frac{\lambda}{2}(R^a - R^b) = \frac{1}{2}(R^a - R^b) - \frac{\lambda}{2}(R^a - R^b) = \left(\frac{1-\lambda}{2}\right)(R^a - R^b).
\]

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\(^1\) Twenty-foot Equivalent Unit (TEU)
\(^ii\) Mediterranean Shipping Company
\(^iii\) China Ocean Shipping Company